DISTRIBUTION LAWS FOR INTEGRABLE EIGENFUNCTIONS

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Abstract. We determine the asymptotics of the joint eigenfunctions of the torus action on a toric Kähler variety. Such varieties are models of completely integrable systems in complex geometry. We first determine the pointwise asymptotics of the eigenfunctions, which show that they behave like Gaussians centered at the corresponding classical torus. We then show that there is a universal Gaussian scaling limit of the distribution function near its center. We also determine the limit distribution for the tails of the eigenfunctions on large length scales. These are not universal but depend on the global geometry of the toric variety and in particular on the details of the exponential decay of the eigenfunctions away from the classically allowed set.

Contents

1. Introduction

A problem of considerable interest in both mathematics and physics is to determine the asymptotics of the distribution functions

$$D_j(t) := \mathrm{Vol}\{z : |\varphi_j(z)|^2 > t\}$$

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of an orthonormal basis $\{\varphi_j\}$ of eigenfunctions of a Laplacian (or similar Hamiltonian) on a compact manifold (see [Y]). In general, it is hopelessly difficult to obtain more than crude bounds on such distribution functions, which of course control the $\mathcal{L}^p$-norms of the eigenfunctions. Numerical analyses and heuristics from quantum chaos and disordered systems suggest however a rich picture in which the asymptotics of $D_j(t)$ is related to the classical dynamics underlying the eigenvalue problem. (Some references will be discussed at the end of the introduction.) The purpose of this paper is to give a rather complete analysis of the limit distribution of eigenfunctions in one of the few settings where such a detailed analysis is possible, namely where the phase space is a toric Kähler variety (M, ω).

Let us recall the definitions (see §1 for details). Toric varieties are complex manifolds on which the complex torus $(\mathbb{C}^*)^m$ acts with an open dense orbit. By a toric Kähler variety we mean a toric variety equipped with a Kähler form (M, ω) that is invariant under the underlying real torus $\mathbf{T}^m$. The action of $\mathbf{T}^m$ is Hamiltonian with respect to ω, and thus toric varieties are models of completely integrable systems. They are of a very special type because integrable systems usually generate an $\mathbb{R}^m$ action rather than a $\mathbf{T}^m$ action. Although there are rigidity theorems limiting the class of such examples in the world of real Riemannian manifolds [LS], toric varieties provide a plentiful collection in the world of complex manifolds.

The torus action can be 'quantized' or linearized on the Hilbert space completion of the coordinate ring

$$\mathcal{H} := \bigoplus_{N=0}^{\infty} H^0(M, L^N), \tag{1}$$

where $L \to M$ is a holomorphic line bundle with $c_1(L) = \frac{1}{2\pi}\omega$ and where $H^0(M, L^N)$ denotes the space of holomorphic sections of its N-th tensor power. This quantization is generally known as the holomorphic (Bargmann-Fock) representation in the physics literature. The space $\mathcal{H}$ is spanned by joint eigenfunctions of the linearized $(\mathbb{C}^*)^m$ action, which we refer to as 'monomials.' In the fundamental case of $M = \mathbb{CP}^m$, the joint eigenfunctions are the monomials given in an affine chart by

$$\chi_\alpha : \mathbb{C}^m \to \mathbb{C}, \quad \chi_\alpha(z) = z^\alpha. \tag{2}$$

The monomials lift (by homogenization) to homogeneous monomials on $\mathbb{C}^{m+1}$.

We consider the case where M is a smooth projective toric variety; i.e., $M = M_P$, $L = L_P$, where P is an integral Delzant polytope (see §2). Then the linearized $\mathbf{T}^m$ action is generated by m commuting operators $\hat{I}_j$, $j = 1, \ldots, m$ on M_P which preserve $H^0(M_P, L_P^N)$, and the joint spectrum of the eigenvalue problem

$$\hat{I}_j \varphi_\alpha^P = \alpha_j \varphi_\alpha^P, \ \ \alpha = (\alpha_1, \ldots \alpha_m) \in \mathbb{R}^m, \quad \varphi_\alpha^P \in H^0(M_P, L_P^N) \tag{3}$$

consists of lattice points $\alpha \in NP \cap \mathbb{Z}^m$.

Our main results concern the asymptotics of their distribution functions

$$D_\gamma(t) := \mathrm{Vol}\{z \in M_P : |\varphi_\gamma^P(z)|^2 > t\} \tag{4}$$

with $\gamma \in NP \cap \mathbb{Z}^m$ as $N \to \infty$. The function $|\varphi_\gamma^P(z)|^2$ is often called the 'Husimi distribution' in the physics literature, and thus our results determine its distribution law. The norm $|\varphi_\gamma^P(z)|$ of $\varphi_\gamma^P(z) \in L_P^N$ is the pull-back of the Fubini-Study norm under a monomial

embedding of the form

$$\Phi_P^c = [c_{\alpha(1)}\chi_{\alpha(1)}, \dots, c_{\alpha(d+1)}\chi_{\alpha(d+1)}] : (\mathbb{C}^*)^m \to \mathbb{CP}^d\,, \qquad P \cap \mathbb{Z}^m = \{\alpha(1), \dots, \alpha(d+1)\}\,, \tag{5}$$

for a choice of constants $c_{\alpha(j)} \in \mathbb{C}^*$. The volume in M_P and the Hermitian norm h_P^c on L_P are by definition the pull-backs of the Fubini-Study metric and form under this monomial embedding, and the $\mathcal{L}^2$ norm on the space $H^0(M, L_P^N)$ is in turn induced from the volume form and the Hermitian pointwise norm of h_P^c. (See §2 for details.)

As Figure 1 illustrates, the monomial φ_α^P is something like a Gaussian bump centered on the real torus $\mu_P^{-1}(\alpha)$, where $\mu_P : M_P \to P$ is the moment map for the classical Hamiltonian $\mathbf{T}^m$-action on M_P (see §1).

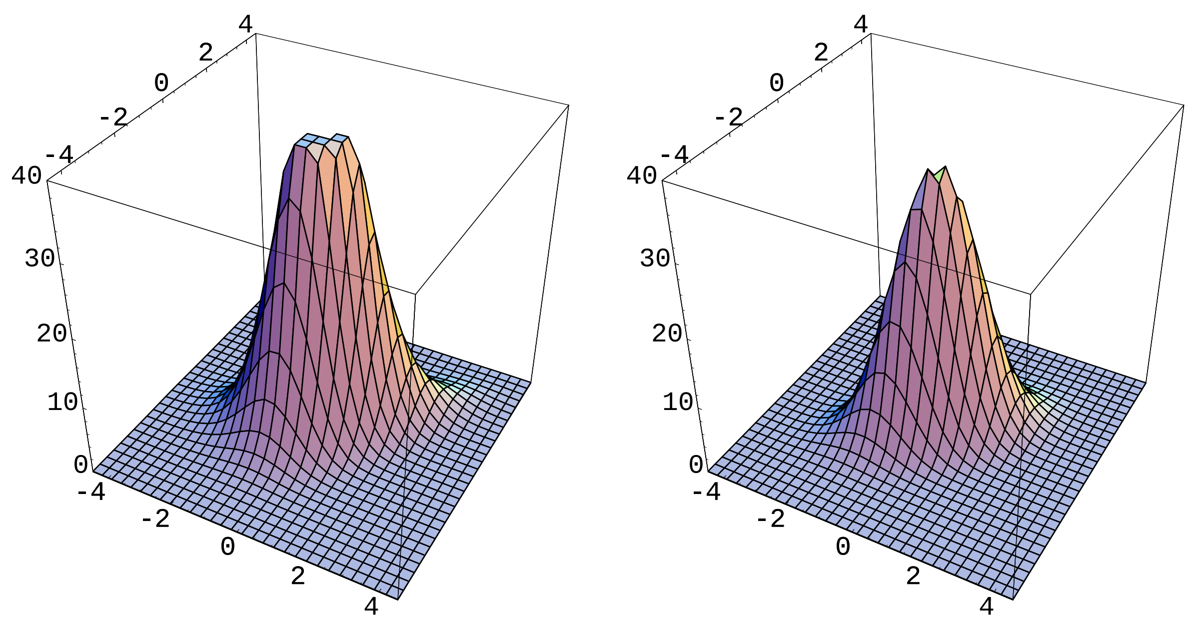

FIGURE 1. $2\pi/N$ times the monomial $|\varphi_{N\alpha}^{7\Sigma}(z)|^2$ for $N = 1$ (left) and $N = \infty$ (right) for $m = 2$ and $\alpha = (2, 3)$, where Σ is the standard simplex. The variable z is chosen as $u \mapsto z = e^{(\rho_\alpha + u/\sqrt{N})/2}$. See Proposition 3.15 in Section 3.

Since we would like to determine the properties of eigenfunctions $\varphi_\alpha \in NP$ when α is large, but not necessarily a multiple of a lattice point in P, we shall consider a sequence of approximate multiples, as in the following definition:

DEFINITION **1.1.** *Let $\alpha_N \in NP \cap \mathbb{Z}^m$ be a sequence of lattice points, and let $x \in P$. We say that $\{\alpha_N\}$ is a sequence of approximate multiples of x if*

$$\alpha_N = Nx + O(1). \tag{6}$$

Our first result gives the pointwise behavior of the eigenfunctions:

THEOREM **1.2.** *Let x be a point in the interior P^o of the polytope P, which is not necessarily a lattice point. Then there exists a non-negative function $b_x^P \in \mathcal{C}^\infty((\mathbb{C}^*)^m)$ such that $b_x^P(z) = 0$*

if and only if $z \in \mu_P^{-1}(x)$, *and for every sequence* $\alpha_N \in NP \cap \mathbb{Z}^m$ *of approximate multiples of* x, *we have*

$$|\varphi^P_{\alpha_N}(z)|^2 = c(P,x)\left(\frac{N}{2\pi}\right)^{m/2} e^{-Nb^P_x(z)}\left[1+O(N^{-1})\right]$$

uniformly on $(\mathbb{C}^*)^m$, *where* $c(P,x) \in \mathbb{R}^+$.

The constant $c(P,x)$ is defined in (43) and (45) (and also appears in Theorems 1.3 and 1.4).

When we consider a sequence of the lattice points of the form $\alpha_N = N\alpha$ with a lattice point α, the assumption that $\alpha \in P^o$ is not necessary. In fact, we will give the pointwise asymptotics for $\alpha_N = N\alpha$ with any lattice point $\alpha \in P \cap \mathbb{Z}^m$ in Section 3 (see Propositions 3.5 and 3.8).

In the above Theorem, the function $b^P_x(z)$ ($x \in P^o$) on $(\mathbb{C}^*)^m$ is defined by

$$b^P_x(z) := \log\left(\frac{\sum_{\beta \in P}|c_\beta z^\beta|^2}{\sum_{\beta\in P} e^{-\langle\, \tau^P_x(z), \beta\,\rangle}|c_\beta z^\beta|^2}\right) - \langle\, \tau^P_x(z), x\,\rangle, \tag{7}$$

where $\tau^P_x(z) \in \mathbb{R}^m$ is the vector given by the equation

$$\mu_P(e^{-\tau^P_x(z)/2}z) = x, \quad z \in (\mathbb{C}^*)^m\,. \tag{8}$$

Here μ_P is the moment map for the $\mathbf{T}^m$ action on M_P (see (23) for the definition), and we write

$$e^r z = (e^{r_1}z_1, \ldots, e^{r_m}z_m) \qquad \text{for } \; r \in \mathbb{R}^m,\; z \in (\mathbb{C}^*)^m\,. \tag{9}$$

We can express (as in [SZ2, (17)–(18)]) the function b^P_x in the more intuitive form as follows: we introduce the real power 'monomials'

$$|\chi_x(z)| := |z|^x = |z_1|^{x_1}\cdots|z_m|^{x_m}$$

and define

$$\mathcal{M}^P_x(z) := \frac{|\chi_x(z)|_P}{\sup|\chi_x(z)|_P}\,, \tag{10}$$

where (cf. (20))

$$|\chi_x(z)|_P := \frac{|\chi_x(z)|}{\sqrt{\sum_{\beta\in P}|c_\beta z^\beta|^2}} \quad (z \in (\mathbb{C}^*)^m).$$

(The normalized monomial $\mathcal{M}^P_x$ has sup-norm 1, attained on the torus $\mu_P^{-1}(x)$.) Then (7) is equivalent to:

$$b^P_x(z) = -2\log \mathcal{M}^P_{q(z)}(z)\,, \tag{11}$$

Since the sequence of monomials flattens out exponentially quickly away from the peak set $\mu_P^{-1}(x)$, the distribution function is clearly tending to zero. The rate of decay of the distribution function is given by the following result:

Theorem **1.3.** (i) *Let* $\alpha_N \in NP \cap \mathbb{Z}^m$ *be a sequence of lattice points which are approximate multiples of* $x \in P^o$ *(see (6)). Then, for* $t > 0$, *we have*

$$D_{\alpha_N}(t) \sim \frac{(\pi m)^{m/2}}{c(P,x)\Gamma(m/2+1)}\left(\frac{\log N}{N}\right)^{m/2}.$$

(ii) *Let $\alpha \in \partial P \cap \mathbb{Z}^m$. Then, for $t > 0$, we have*

$$D_{N\alpha}(t) \sim \frac{(\pi d(\alpha))^{d(\alpha)/2}}{c(P,\alpha)\Gamma(d(\alpha)/2+1)} \left(\frac{\log N}{N}\right)^{d(\alpha)/2},$$

where $d(\alpha) := m + \operatorname{codim} F_\alpha$, F_α being the face of P containing α.

Here $\sim$ means the ratio of the left and right hand sides tends to 1, and the constants $c(P,x)$, $c(P,\alpha)$ are given by (43) and (70)–(45). We recall that if x is a point in a face F of codimension r, then $\mu_P^{-1}(x) \cong \mathbf{T}^{m-r}$. Hence $d(\alpha) = 2m - \dim \mu_P^{-1}(\alpha)$.

The exponentially localized behavior of the monomials suggests studying the distribution function on various length scales. First, we show that the D_{α_N} have a universal scaling limit on a small length scale:

THEOREM **1.4.** (i) *Let $\alpha_N \in NP \cap \mathbb{Z}^m$ be a sequence of lattice points satisfying the condition* (6) *with some point $x \in P^o$. Then, for $0 < t \leq c(P,x)$, we have*

$$\lim_{N\to\infty} (N/2\pi)^{m/2} D_{\alpha_N}\left((N/2\pi)^{m/2} t\right) = \frac{1}{c(P,x)\Gamma(m/2+1)} (\log(c(P,x)/t))^{m/2}.$$

(ii) *Let $\alpha \in P \cap \mathbb{Z}^m$. Then*

$$\lim_{N\to\infty} (N/2\pi)^{d(\alpha)/2} D_{N\alpha}\left((N/2\pi)^{d(\alpha)/2} t\right) = \frac{1}{c(P,\alpha)\Gamma(d(\alpha)/2+1)} (\log(c(P,\alpha)/t))^{d(\alpha)/2},$$

for $0 < t \leq c(P,\alpha)$, where $d(\alpha) = \operatorname{codim} \mu_P^{-1}(\alpha)$.

A sample graph of the scaling limit distribution function for $P = 7\Sigma$ is given in Figure 2.

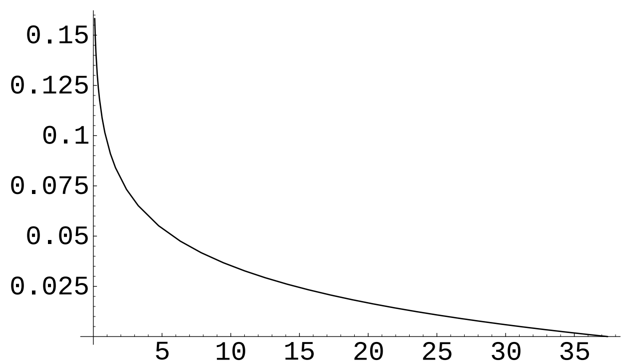

FIGURE 2. Scaling limit distribution for $|\varphi_{N\alpha}^P|^2$ with $m = 2$, $\alpha = (2,3)$, $P = 7\Sigma$

As in Theorem 1.4, the limit of the rescaled distributions has a universal form, *i.e.* it does not depend on the geometry of the manifold M_P, and is given by a logarithmic power of the form $(\log c/t)^{d/2}$ with some constant d. The logarithmic power appears because the $\mathcal{L}^2$-normalized monomials are close to Gaussian around the peak set $\mu_P^{-1}(x)$ on a vector space of dimension $m/2$ (or dimension $d(\alpha)$ for the boundary lattice cases). More precisely, Theorem 1.2 (and Propositions 3.5 and 3.8) shows that the function b_x^P has a positive definite Hessian at the peak point. We then observe that the distribution of a Gaussian function on $\mathbb{R}^d$,

$$g(u) := \frac{e^{-\langle Au,u\rangle/2}}{\sqrt{\det A}}, \quad u \in \mathbb{R}^d,$$

(where A is a real positive $(d \times d)$-matrix) is given by the logarithmic power law

$$\nu_A(u \in \mathbb{R}^d\,;\, g(u) > t) = \frac{1}{c\Gamma(d/2+1)} \left(\log \frac{c}{t}\right)^{d/2}$$

$$c = \frac{1}{\sqrt{\det A}}, \quad 0 < t \leq c,$$

relative to the normalized Lebesgue measure

$$\nu_A := \frac{\det A}{(2\pi)^{d/2}}\, du.$$

Thus, the rescaled distribution of an $\mathcal{L}^2$-normalized monomial at the 'center' of its localized bump has a universal Gaussian form.

To analyze the 'tails' of the eigenfunctions, we next use an exponential rescaling of the distribution function so that the global distribution law has a non-zero limit as $N \to \infty$. As may be expected, it is no longer universal but depends on the geometry of (M_P, ω_P).

THEOREM **1.5.** *Let $\alpha_N \in NP \cap \mathbb{Z}^m$ satisfy the condition* (6) *with a point $x \in P^o$. Then*

$$\lim_{N\to\infty} D_{\alpha_N}(e^{-Nt}) = \int_{\{\rho \in \mathbb{R}^m\,;\, b_x^P(\rho) < t\}} \det A(\rho)\, d\rho\,,$$

where $A(\rho)$ is the Hessian matrix of $\log \sum_{\beta\in P} |c_\beta|^2 e^{\langle \beta, \rho\rangle}$.

A more general scaling limit law is given in Theorem 4.3. In the above theorem, the assumption that $x \in P^o$ is not necessary. In fact, we will give similar result for $\alpha_N = N\alpha$ with any lattice point $\alpha \in P$ (Theorem 4.3) in Section 4.

Our strategy for proving Theorems 1.3 and 1.5 on the distribution functions of monomials is based on their pointwise asymptotics (Theorem 1.2, and also Propositions 3.5 and 3.8 in Section 3). Pointwise asymptotics of monomials are more or less equivalent to asymptotics of their L^{2k} norms. Since the latter are of independent interest, we state these asymptotics explicitly in:

THEOREM **1.6.** (i) *Let $\alpha_N \in NP \cap \mathbb{Z}^m$ satisfy the condition* (6) *for a point $x \in P^o$. Let $\|\varphi_\gamma^P\|_{2k}$ denote the $\mathcal{L}^{2k}$-norm of the $\mathcal{L}^2$-normalized monomial φ_γ^P with the weight $\gamma \in NP$. Then we have*

$$\|\varphi_{\alpha_N}\|_{2k}^{2k} = \frac{c(P,x)^{k-1}}{k^{m/2}} \left(\frac{N}{2\pi}\right)^{(k-1)m/2} (1 + O_k(N^{-1})),$$

where $O_k(N^{-1})$ depends on k.

(ii) *Let $\alpha \in P \cap \mathbb{Z}^m$ with $d(\alpha) = \operatorname{codim} \mu_P^{-1}(\alpha)$. Then we have*

$$\|\widehat{\varphi}_{N\alpha}^P\|_{2k}^{2k} = \frac{c(P,\alpha)^{k-1}}{k^{d(\alpha)/2}} \left(\frac{N}{2\pi}\right)^{(k-1)d(\alpha)/2} (1 + O_k(N^{-1})).$$

We close the introduction with some general remarks and references. As mentioned above, our results pertain to phase space distribution of eigenfunctions (Husimi distributions) rather than to their configuration space distribution. To our knowledge, the only prior example in dimension > 1 for which the limit distribution of eigenfunctions has been determined is the case of certain (so-called) Hecke eigenfunctions of discrete quantum cat maps, due to

Kurlberg-Rudnick [KR]. They work in a simpler discrete model rather than the holomorphic model. The main result of Kurlberg-Rudnick [KR] is that the distribution functions of (un-scaled) eigenfunctions tend to the semi-circle law. Their method was to relate the eigenfunctions to exponential sums studied by Katz [Ka] and to apply the value distribution of exponential sums. Since value distribution depends on the representation, it is not clear that the same semi-circle law would hold for Hecke eigenfunctions in the holomorphic (Bargmann-Fock) representation, and this appears to be a challenging and interesting problem. Part of the motivation for this paper was to set a baseline for eigenfunction distribution problems by studying a class of explicitly solvable examples.

It would also be interesting to study the limit distribution of real eigenfunctions in the Schrödinger representation, i.e. on the configuration space rather than the phase space. To our knowledge, the physics results mainly pertain to these configuration space results. The cases most studied and speculated about are those of chaotic or disordered systems. When eigenfunctions are delocalized, their spatial distribution is conjectured to be Gaussian (see e.g. [Be, FE, He, HR, Mi, SS, PA]). In the opposite regime where the eigenfunctions of disordered systems are exponentially localized, the expected distribution in the low amplitude (tail) region is given by a power of a logarithm [MF], precisely the one we obtained in the high amplitude (center) region. The reason is that the distribution law is universal and Gaussian in the tail region for exponentially localized eigenfunctions of disordered systems [MF], while it is universal and Gaussian in the center for our problem (Theorem 1.2).

The only studies we have located which are related to distribution laws of integrable eigenfunctions are those of one of the authors with J. A. Toth (cf. [TZ]) and that of Berry-Hannay-Ozorio de Almeida [BHO], which describes the asymptotic expansions of $\mathcal{L}^{2p}$ norms (moment intensities) of real oscillatory integrals of several stable types. Such oscillatory integrals define quasimodes for a quantum integrable system, and in generic cases one can express eigenfunctions as oscillatory integrals of various kinds [TZ]. In general, many possible kinds of oscillatory integrals could arise, including ones associated to singular Lagrangean tori. Hence, one can only expect complete results in special cases. To take the simplest example, our methods could be adapted to find the scaling limit distributions of squares of the standard spherical harmonics Y_m^N on S^2 (or S^m). These eigenfunctions are also joint eigenfunctions of commuting operators which generate a quantum torus action. To our knowledge, the distribution laws of even such simple eigenfunctions are unknown at this time.

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2. Background on toric varieties and moment polytopes

We summarize here some basic facts and terminology on toric varieties from [STZ]. Recall that a *toric variety* is a complex algebraic variety M containing the complex torus

$$(\mathbb{C}^*)^m := (\mathbb{C} \setminus \{0\}) \times \cdots \times (\mathbb{C} \setminus \{0\})$$

as a Zariski-dense open set such that the group action of $(\mathbb{C}^*)^m$ on itself extends to M. We consider here smooth projective toric varieties; they can be given the structure of a

symplectic manifold such that the restriction of the action to the underlying real torus

$$\mathbf{T}^m = \{(\zeta_1, \dots \zeta_m) \in (\mathbb{C}^*)^m : |\zeta_j| = 1, 1 \le j \le m\}$$

is a Hamiltonian action (see §2.2). These toric varieties can be constructed from Delzant polytopes either by symplectic reduction (see [Gu]) or by gluing affine toric varieties described by the normal fan of the polytope (see [Fu]). However, for our analysis, it is more convenient to define the toric variety M_P associated to the Delzant polytope P through a *monomial embedding* as follows (see [GKZ, Chapter 5]). Suppose that P is a Delzant polytope and let

$$P \cap \mathbb{Z}^m = \{\alpha(1), \alpha(2), \dots \alpha(d+1)\} .$$

We shall write $\sum_{\alpha \in P} = \sum_{\alpha \in P \cap \mathbb{Z}^m}$. (Recall that a *Delzant polytope* is a convex integral polytope in $\mathbb{R}^m$ with the property that each vertex is incident to exactly m edges and the primitive vectors in $\mathbb{Z}^m$ parallel to these edges generate $\mathbb{Z}^m$.)

To define the monomial embedding, we fix an arbitrary $c = (c_{\alpha(1),\dots,\alpha(d+1)}) \in (\mathbb{C}^*)^{d+1}$. Then, we define

$$\Phi_P^c = \left[c_{\alpha(1)}\chi_{\alpha(1)}, \dots, c_{\alpha(d+1)}\chi_{\alpha(d+1)}\right] : (\mathbb{C}^*)^m \to \mathbb{CP}^d ; \tag{12}$$

i.e.,

$$\Phi_P^c(z) = [c_{\alpha(1)} z^{\alpha(1)}, \dots, c_{\alpha(d+1)} z^{\alpha(d+1)}], \quad z \in (\mathbb{C}^*)^m.$$

The toric variety $M_P^c = M_P$ is defined as the Zariski-closure of the image $\Phi_P^c((\mathbb{C}^*)^m)$ of the monomial embedding Φ_P^c in the complex projective space $\mathbb{CP}^d$.

Since our polytope P is assumed to be Delzant, Φ_P^c is an embedding and the variety M_P is smooth. The symplectic (or Kähler) form on M_P is given by

$$\omega_P^c = \omega_P := \Phi_P^{c*}\omega_{\mathrm{FS}} , \tag{13}$$

where $\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\bar\partial \log \|\zeta\|^2$ denotes the Fubini-Study Kähler form on $\mathbb{CP}^d$ with homogeneous coordinates $(\zeta_0, \dots, \zeta_d)$. On $(\mathbb{C}^*)^m$, we have

$$\omega_P^c = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \sum_{\alpha \in P} |c_\alpha|^2 |z^\alpha|^2. \tag{14}$$

The volume form on M_P is given by $d\mathrm{Vol}_M = \frac{1}{m!}\omega_P^m$.

2.1. **The line bundle L_P^c and associated circle bundle X_P^c.** We define the line bundle $L_P^c \to M_P^c$ by $L_P = L_P^c := \Phi_P^{c*}\mathcal{O}(1)$, where $\mathcal{O}(1)$ denotes the hyperplane section bundle on $\mathbb{CP}^d$. Recall that the space of holomorphic sections $H^0(\mathbb{CP}^d, \mathcal{O}(1))$ consists of the linear functions $\lambda : \mathbb{C}^{d+1} \to \mathbb{C}$, and that the *Fubini-study metric* on $\mathcal{O}(1)$ is given by

$$|\lambda|_{\mathrm{FS}}([\zeta]) = \frac{|\lambda(\zeta)|}{\|\zeta\|} \qquad (\zeta \in \mathbb{C}^{d+1}) ,$$

which has curvature form $\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\bar\partial \log \|\zeta\|^2$. We endow L_P^c with the Hermitian metric $h_P^c := \Phi_P^{c*} h_{\mathrm{FS}}$, which has curvature ω_P^c.

Each monomial χ_α with $\alpha \in P \cap \mathbb{Z}^m$ corresponds to a section of $H^0(M_P^c, L_P^c)$ and vice versa. To explicitly define this correspondence, we make the identifications:

$$\chi_{\alpha(j)}^P \equiv c_{\alpha(j)}^{-1}\Phi_P^{c*}\zeta_j \in H^0(M_P^c, L_P^c) \cong \Phi_P^{c*}H^0(\mathbb{CP}^d, \mathcal{O}(1)) , \quad 1 \le j \le d+1 . \tag{15}$$

More generally,

$$\Phi_P^{c*}H^0(\mathbb{CP}^d, \mathcal{O}(N)) \cong H^0(M_P, L_P^N) \qquad (N \ge 1), \tag{16}$$

and a basis for $H^0(M_P, L_P^N)$ is given by the sections $\{\chi_\gamma^P : \gamma \in NP \cap \mathbb{Z}^m\}$ corresponding to the monomials $\{\chi_\gamma\}$. These sections are given by

$$\chi_\gamma^P = \chi_{\beta_1}^P \otimes \cdots \otimes \chi_{\beta_N}^P,$$

where $\beta_1, \ldots, \beta_N \in P \cap \mathbb{Z}^m$ such that $\gamma = \beta_1 + \cdots + \beta_N$ (see[Fu, STZ]).

So far, we have not specified the constants c_α. For studying our phenomena, the choice of constants defining the toric variety M_P is not important. However, when our polytope P is the full simplex $p\Sigma$, we shall use the special choice $c_\alpha = \binom{p}{\alpha}^{1/2}$, where $\binom{p}{\alpha}$ is the multinomial coefficient (see §2.4).

The associated principal S^1-bundle $X_P^c = X_P$ of the line bundle $L_P \to M_P$ is defined by

$$X_P^c := \{(z, v) \in L_P^{-1}\,;\ |v|_P = 1\},$$

where $|v|_P$ denotes the norm of v with respect to the Hermitian metric on L_P^{-1} induced by h_P^c.

We identify sections s_N of L^N with equivariant functions $\widehat{s}_N$ on X by the rule

$$\widehat{s}_N(\lambda) = \left(\lambda^{\otimes N}, s_N(z)\right)\,, \quad \lambda \in X_z\,. \tag{17}$$

Clearly, $\widehat{s}_N(e^{i\theta} \cdot x) = e^{iN\theta}\widehat{s}_N(x)$ if $s_N \in H^0(M_P^c, L_P^N)$. It should be noted that for each $s \in H^0(M_P^c, L_P^N)$, we have

$$|\widehat{s}_N(x)| = |s_N(z)|_P$$

where $x \in X_P$ is in the fiber over $z \in M_P^c$, $|s_N(z)|_P$ denotes the norm with respect to the Hermitian metric on L_P^N induced by the metric h_P^c.

In particular, for $\alpha \in P \cap \mathbb{Z}^m$, the 'monomial' $\chi_\alpha^P \in H^0(M_P^c, L_P^c)$ given by (15) lifts to an equivariant function $\widehat{\chi}_\alpha^P$ on the circle bundle $X_P^c \to M_P^c$, and we write

$$\widehat{m}_{\alpha(j)}^P := c_{\alpha(j)}\widehat{\chi}_{\alpha(j)}^P = \zeta_j \circ \iota_P \tag{18}$$

where $\iota_P : X_P^c \to S^{2d+1}$ is the lift of the embedding $M_P^c \hookrightarrow \mathbb{CP}^d$ ($d = \#P - 1$), that is, ι_P is the restriction to X_P of the natural inclusion $L_P^{-1} \hookrightarrow \mathcal{O}(-1)$. (Of course, $\widehat{m}_\alpha^P$ depends on c, which we omit to simplify notation.) We also consider the monomials

$$m_\alpha^P := c_\alpha \chi_\alpha^P$$

so that $\widehat{m}_\alpha^P$ is the equivariant lift of m_α to X_P^c. In terms of local coordinates (z, θ) on $\pi^{-1}((\mathbb{C}^*)^m) \subset X_P^c$, we have

$$\widehat{m}_\alpha^P(z, \theta) = \frac{e^{i\theta} c_\alpha z^\alpha}{\left(\sum_{\beta \in P} |c_\beta z^\beta|^2\right)^{1/2}}\,. \tag{19}$$

Since its absolute value is independent of θ, we can write

$$|m_\alpha^P(z)|_P = |\widehat{m}_\alpha^P(z)| = \frac{|c_\alpha z^\alpha|}{\left(\sum_{\beta \in P} |c_\beta z^\beta|^2\right)^{1/2}}\,. \tag{20}$$

We give $H^0(M_P, L_P^N)$ the inner product

$$\langle s_1, \bar{s}_2 \rangle = \int_M \left\langle s_1(z), \overline{s_2(z)} \right\rangle_{h_N} d\mathrm{Vol}_M(z)\,, \qquad s_1, s_2 \in H^0(M_P, L_P^N)\,, \tag{21}$$

and the $\mathcal{L}^2$ norm $\|s\| = \langle s, \bar{s}\rangle$. We note that the sections

$$\left\{\chi_\alpha^P \in H^0(M_P, L_P^N) : \alpha = (\alpha_1, \dots, \alpha_m) \in NP\right\}$$

are orthogonal but not normalized. We normalize them to obtain an orthonormal basis for $H^0(M_P, L_P^N)$ consisting of the sections

$$\varphi_\alpha^P := \frac{\chi_\alpha^P}{\|\chi_\alpha^P\|} \ . \tag{22}$$

Their equivariant lifts $\widehat{\varphi}_\alpha^P$ form an orthonormal set of monomials on X_P^c.

We include below a table of notation to help the reader keep track of the various monomials:

	monomials on $\mathbb{C}^m$	sections of L_P^N	monomials on X_P
$\alpha \in NP$	$\chi_\alpha(z) = z^\alpha$	χ_α^P	$\widehat{\chi}_\alpha^P$
$\alpha \in P$		$m_\alpha^P = c_\alpha \chi_\alpha^P \quad (N=1)$	$\widehat{m}_\alpha^P$
$\alpha \in NP$		$\varphi_\alpha^P = \chi_\alpha^P / \|\chi_\alpha^P\|$	$\widehat{\varphi}_\alpha^P$

2.2. **Moment maps and torus actions.** The group $(\mathbb{C}^*)^m$ acts on M_P^c and the subgroup $\mathbf{T}^m$ acts in a Hamiltonian fashion. Let us recall the formula for its moment map $\mu_P^c : M_P^c \to \mathbb{R}^m$: on the open orbit $(\mathbb{C}^*)^m$, we have

$$\mu_P(z) = \mu_P^c(z) = \frac{1}{\sum_{\alpha \in P} |c_\alpha|^2 |z^\alpha|^2} \sum_{\alpha \in P} |c_\alpha|^2 |z^\alpha|^2 \alpha = \sum_{\alpha \in P} |\widehat{m}_\alpha^P(z)|^2 \alpha \ . \tag{23}$$

For any c, the image of M_P^c under μ_P^c equals P. The moment map μ_P is invariant under the $\mathbf{T}^m$-action on M_P. By the identification $(\mathbb{C}^*)^m \cong \mathbf{T}^m \times \mathbb{R}^m$, the moment map μ_P defines a map from $\mathbb{R}^m$ to P, which is a diffeomorphism between $\mathbb{R}^m$ and the interior P^o of P.

The action of the real torus $\mathbf{T}^m$ lifts from M_P^c to X_P^c and combines with the S^1 action to define a $\mathbf{T}^{m+1}$ action on X_P^c. Recall that under the monomial embedding

$$\Phi_P^c : (\mathbb{C}^*)^m \hookrightarrow M_P^c \hookrightarrow \mathbb{CP}^d, \qquad z \mapsto \left[c_{\alpha(1)} z^{\alpha(1)}, \dots, c_{\alpha(d+1)} z^{\alpha(d+1)}\right] ,$$

the $\mathbf{T}^m$ action on $M_P^c \subset \mathbb{CP}^d$ is given by

$$e^{i\varphi} \cdot [\zeta_1, \dots, \zeta_{d+1}] = \left[e^{i\langle \alpha(1), \varphi\rangle} \zeta_1, \dots, e^{i\langle \alpha(d+1), \varphi\rangle} \zeta_{d+1}\right] . \tag{24}$$

The action (24) lifts to an action on L_P^{-1}:

$$e^{i\varphi} \cdot \zeta = \left(e^{i\langle \alpha(1), \varphi\rangle} \zeta_1, \dots, e^{i\langle \alpha(d+1), \varphi\rangle} \zeta_{d+1}\right) . \tag{25}$$

Since the circle bundle $X_P^c \subset S^{2d+1}$ is invariant under this action, (25) also gives a lift of the action (24) to X_P^c.

We also have the standard circle action on X_P^c:

$$e^{i\theta} \cdot \zeta = e^{i\theta} \zeta \ , \tag{26}$$

which commutes with the $\mathbf{T}^m$-action (25). Combining (25) and (26), we then obtain a $\mathbf{T}^{m+1}$-action on X_P^c:

$$(e^{i\theta}, e^{i\varphi_1}, \dots, e^{i\varphi_m}) \bullet \zeta = e^{i\theta}(e^{i\varphi} \cdot \zeta) \ . \tag{27}$$

2.3. **Fourier analysis.** In this section, we shall explain an aspect of Fourier analysis on toric varieties which describe the complete integrability of the system.

The Hardy space $\mathcal{H}^2(X_P^c)$ is the Hilbert space spanned by the equivariant lifts of sections to functions on X_P^c with the inner product

$$\langle f, \bar{g} \rangle = \int_X f \bar{g} dV, \quad dV = \alpha_P \wedge (d\alpha_P)^{n-1},$$

where α_P is a contact 1-form defined by the Hermitian connection on L_P^{-1} such that $d\alpha_P = \pi^* \omega_P$. Under the identification $\mathcal{H} \cong \mathcal{H}^2(X_P^c)$, the inner product is the same as the inner product on $\mathcal{H} = \bigoplus_N H^0(M_P, L_P^N)$ given by (21). Alternately, $\mathcal{H}^2(X_P^c)$ consists of the functions $F \in \mathcal{L}^2(X_P^c)$ satisfying $\bar{\partial}_b F = 0$ (see e.g., [SZ1, Ze]).

Under the S^1 action, the Hardy space then has the orthogonal decomposition

$$\mathcal{H}^2(X_P^c) = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^2(X_P^c) , \tag{28}$$

where $\mathcal{H}_N^2(X_P^c)$ consists of elements $\hat{s} \in \mathcal{H}^2(X_P^c)$ such that $\hat{s}(e^{i\theta} \cdot x) = e^{iN\theta} \hat{s}(x)$. We recall that the (equivariant) 'Szegö projectors' Π_N are the orthogonal projection onto $H^0(M_P^c, L_P^{cN}) \cong \mathcal{H}_N^2(X_P^c)$. If $\{S_j^N\}$ denotes an orthonormal basis of $H^0(M_P^c, L_P^{cN})$, and $\widehat{S}_j^N$ denote their lifts to X, then the projector Π_N is given by the kernel function

$$\Pi_N(x, y) = \sum_{j=1}^{k_N} \widehat{S}_j^N(z) \overline{\widehat{S}_j^N(y)} \ : \mathcal{L}^2(X_P^c) \to \mathcal{H}_N^2(X_P^c). \tag{29}$$

We now describe how one can combine the eigenvalue problems given by (3) for varying N into a homogeneous scalar eigenvalue problem on X. To do this, we define for each $N \in \mathbb{N}$, the 'homogenization' $\widehat{NP} \subset \mathbb{Z}^{m+1}$ of the lattice point in the polytope NP to be the set of all lattice point $\widehat{\alpha}^N$ of the form

$$\widehat{\alpha}^N = \widehat{\alpha} := (Np - |\alpha|, \alpha_1, \ldots, \alpha_m), \quad \alpha = (\alpha_1, \ldots, \alpha_m) \in NP \cap \mathbb{Z}^m,$$

where $p \geq \max_{\beta \in P \cap \mathbb{Z}^m} |\beta|$. We also define the cone $\Lambda_P = \bigcup_{N=1}^{\infty} \widehat{NP}$. It is well known that rays $\mathbb{N}\widehat{\alpha}$ in this cone define a semiclassical limit.

In this section, we use the more precise notation $\widehat{\varphi}_{\hat{\alpha}}^P(x)$ for the $\mathcal{L}^2$-normalized monomial $\widehat{\varphi}_{\alpha}^P(x)$ (since N is not specified in the latter), for $\widehat{\alpha} \in \Lambda_P$.

The torus action on X_P^c can be quantized to define an action of the torus as unitary operators on $\mathcal{H}^2(X_P^c)$. Specifically, we let $\hat{I}_1, \ldots \hat{I}_m$ denote the differential operators on X_P^c generated by the $\mathbf{T}^m$ action:

$$(\hat{I}_j \hat{S})(\zeta) = \frac{1}{i} \frac{\partial}{\partial \varphi_j} \hat{S}(e^{i\varphi} \cdot \zeta)|_{\varphi=0} \ , \quad \hat{S} \in \mathcal{C}^\infty(X_P^c) \ . \tag{30}$$

We recall the following observation from [STZ]:

Proposition 2.1. *For $1 \leq j \leq m$,*

(i) $\hat{I}_j : \mathcal{H}_N^2(X_P^c) \to \mathcal{H}_N^2(X_P^c)$;

(ii) *The lifted monomials $\widehat{\varphi}_{\hat{\alpha}}^P \in \mathcal{H}_N^2(X_P^c)$ satisfy $\hat{I}_j \widehat{\varphi}_{\hat{\alpha}}^P = \alpha_j \widehat{\varphi}_{\hat{\alpha}}^P$ ($\widehat{\alpha} \in \Lambda_P$).*

Furthermore, we note that

$$\frac{\partial}{\partial\theta} : \mathcal{H}^2_N(X^c_P) \to \mathcal{H}^2_N(X^c_P) \ , \qquad \frac{1}{i}\frac{\partial}{\partial\theta}\hat{s}_N = N\hat{s}_N \quad \text{for} \ \ \hat{s}_N \in \mathcal{H}^2_N(X^c_P) \ . \tag{31}$$

Thus, the monomials $\widehat{\varphi}_{\hat{\alpha}}$ are the solutions of the joint eigenvalue problem

$$\hat{I}_j\widehat{\varphi}_{\hat{\alpha}} = \widehat{\alpha}_j\widehat{\varphi}_{\hat{\alpha}}, \ \ \widehat{\alpha} \in \mathbb{R}^{m+1}, \quad \bar{\partial}_b\widehat{\varphi}_{\hat{\alpha}} = 0, \ \ j = 0, \dots, m \tag{32}$$

with the commuting operators:

$$\hat{I}_0 = \frac{p}{i}\frac{\partial}{\partial\theta} - \sum_{j=1}^m \hat{I}_j, \quad \hat{I}_1, \ \dots \ , \ \hat{I}_m \ . \tag{33}$$

The joint eigenvalues are the lattice points $\widehat{\alpha} \in \Lambda_P$.

2.4. **Monomials on projective space.** In the case of $\mathbb{CP}^m$, the $\mathcal{L}^q$-norms of the monomials $\widehat{\varphi}^{p\Sigma}_{N\alpha}$ can be evaluated explicitly in an elementary way. We give the details in this section.

When the polytope P is the unit simplex Σ, we have $M_\Sigma = \mathbb{CP}^m$; furthermore L_Σ is the hyperplane bundle $\mathcal{O}(1)$. We can identify $L_\Sigma^{-1} = \mathcal{O}_{\mathbb{CP}^m}(-1)$ with $\mathbb{C}^{m+1}$ with the origin blown up, and the circle bundle $X_\Sigma \subset L_\Sigma^{-1}$ is identified with the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$. The equivariant lifts to X_Σ of sections of $\mathcal{O}(N) = L_\Sigma^N$ consist of homogeneous polynomials

$$F(\zeta_0, \dots, \zeta_m) = \sum_{|\lambda|=N} C_\lambda \zeta^\lambda \qquad (\zeta^\lambda = \zeta_0^{\lambda_0} \cdots \zeta_m^{\lambda_m})$$

in $m+1$ variables. The induced Fubini-Study metric on $\mathcal{O}(N)$ is given by

$$|F(\zeta)|_\Sigma = |F(\zeta)|/\|\zeta\|^N, \quad \text{for} \ \ F \in H^0(\mathbb{CP}^m, \mathcal{O}(N)) \ .$$

(Here the constants c_α are taken to be 1.) Identifying F with the polynomial $f(z) = F(1, z_1, \dots, z_m)$, the norm can be written

$$|f(z)|_\Sigma = |f(z)|/(1 + \|z\|^2)^{N/2} \qquad (z \in \mathbb{C}^m) \ ,$$

and the inner product is given by:

$$\langle f, \bar{g}\rangle = \frac{1}{m!}\int_{\mathbb{C}^m} \frac{\langle f(z), \overline{g(z)}\rangle}{(1 + \|z\|^2)^p}\,\omega^m_{\mathrm{FS}}(z), \quad f, g \in H^0(\mathbb{CP}^m, \mathcal{O}(p)). \tag{34}$$

We are interested here in the case $P = p\Sigma$. Then $M_{p\Sigma} = \mathbb{CP}^m$, and the line bundle $L_{p\Sigma}$ is identified with the p-th tensor power $\mathcal{O}(p)$ of the hyperplane section bundle. The circle bundle $X_{p\Sigma}$ is the lens space $X_{p\Sigma} = S^{2m+1}/\mathbb{Z}_p$. By lifting equivariant functions from the lens space $X_{p\Sigma} = S^{2m+1}/\mathbb{Z}_p$ to the sphere S^{2m+1}, we see that $\mathcal{H}^2_N(X_{p\Sigma}) \cong \mathcal{H}^2_{Np}(S^{2m+1})$. Hence, we shall replace the N-th Hardy space $\mathcal{H}^2_N(X_{p\Sigma})$ by $\mathcal{H}^2_{Np}(S^{2m+1})$ below. Then the equivariant lift $\widehat{\chi}^{p\Sigma}_\alpha : S^{2m+1} \to \mathbb{C}$ of $\chi^{p\Sigma}_\alpha \in H^0(\mathbb{CP}^m, \mathcal{O}(p))$ is given by the homogenization:

$$\widehat{\chi}^{p\Sigma}_\alpha(x) = x^{\widehat{\alpha}}, \quad \widehat{\alpha} = (p - |\alpha|, \alpha_1, \dots, \alpha_m).$$

In this case, we shall use the special choice of the coefficients of the monomial embedding:

$$c^*_\alpha = \binom{p}{\alpha}^{\frac{1}{2}}, \qquad \binom{p}{\alpha} := \frac{p!}{(p - |\alpha|)!\alpha_1! \cdots \alpha_m!} \ ,$$

so that by (14), we have

$$\omega_{p\Sigma}^{c^*} = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log\left(\sum_{|\alpha|\le p}\binom{p}{\alpha}|z^\alpha|^2\right) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log(1+\|z\|^2)^p = p\omega_{\rm FS}.$$

Furthermore,

$$\mu_{p\Sigma}(z) := \mu_{p\Sigma}^{c^*}(z) = \frac{1}{\sum_{|\alpha|\le p}\binom{p}{\alpha}|z^\alpha|^2}\sum_{|\alpha|\le p}\binom{p}{\alpha}|z^\alpha|^2\alpha = \frac{p}{1+\sum|z_j|^2}(|z_1|^2,\dots,|z_m|^2)\,, \quad (35)$$

where the last equality follows by differentiating the identity $(1+\sum x_j)^p = \sum_{|\alpha|\le p}\binom{p}{\alpha}x^\alpha$. Note that this choice gives us the scaling formula

$$\mu_{p\Sigma} = p\mu_\Sigma.$$

As before, for each m-dimensional multi-index β with $|\beta|\le p$, we define an $(m+1)$-dimensional multi-index $\widehat\beta$ by

$$\widehat\beta = (\widehat\beta_0,\widehat\beta_1,\dots,\widehat\beta_m), \quad \widehat\beta_0 = p-|\beta|, \quad \widehat\beta_j=\beta_j \quad (j=1,\dots,m).$$

Recall that $|\widehat\varphi_\beta^{p\Sigma}(x)| = |\varphi_\beta^{p\Sigma}(z)|_{p\Sigma}$ with $x\in S^{2m+1}$, $\pi(x)=z\in\mathbb{CP}^m$.

The $\mathcal{L}^q$-norms of the monomial $\widehat\varphi_{N\alpha}^{p\Sigma}$ can be evaluated explicitly as follows.

Proposition **2.2.** *For $\alpha\in p\Sigma\cap\mathbb{Z}^m$, we have the precise formula:*

$$\|\widehat\varphi_{N\hat\alpha}^{p\Sigma}\|_q^q = \left[\frac{(Np+m)!}{(N\widehat\alpha)!}\right]^{q/2}\frac{\prod_{j=0}^m\Gamma(Nq\widehat\alpha_j/2+1)}{p^{m(q/2-1)}\Gamma(Npq/2+m+1)}.$$

Proof. We write $\widehat\chi_{N\hat\alpha} = \widehat\chi_{N\hat\alpha}^{p\Sigma}$, which we consider as a function in $\mathcal{H}^2_{Np}(S^{2m+1})$. For $q\ge 1$, we set

$$\mathcal{I}_q(N) = \int_{\mathbb{C}^{m+1}} e^{-|x|^2}|\widehat\chi_{N\hat\alpha}(x)|^q\,d\ell(x),$$

where $d\ell(x)$ denotes Lebesgue measure on $\mathbb{C}^{m+1}$. We shall compute the integral $\mathcal{I}_q(N)$ in two ways. First we use polar coordinates on $\mathbb{C}^{m+1}$. The measure $d\ell$ is expressed as

$$d\ell(x) = r^{2m+1}\,dr\,d\sigma, \quad x=r\sigma, \quad r>0, \quad \sigma\in S^{2m+1}.$$

Then we have

$$\begin{aligned}\mathcal{I}_q(N) &= \int_0^\infty e^{-r^2}r^{Npq+2m+1}\,dr\int_{S^{2m+1}}|\widehat\chi_{N\hat\alpha}(\sigma)|^q\,d\sigma\\ &= \frac12\Gamma(Npq/2+m+1)\int_{S^{2m+1}}|\widehat\chi_{N\hat\alpha}(\sigma)|^q\,d\sigma\,.\end{aligned}$$

To relate the volume form $d\sigma$ on S^{2m+1} to that on $X_{p\Sigma}$, we recall that $\omega_{p\Sigma} = p\omega_{\rm FS}$ and therefore the volume form on $X_{p\Sigma}$ is p^m times the Fubini-Study volume. Recalling our convention that $\mathrm{Vol}(X_\Sigma) = \mathrm{Vol}(\mathbb{CP}^m) = \frac{1}{m!}$, we then have $d\mathrm{Vol}_{X_{p\Sigma}} = \frac{p^m}{2\pi^{m+1}}\,d\sigma$, and hence

$$\mathcal{I}_q(N) = \frac{\pi^{m+1}}{p^m}\Gamma(Npq/2+m+1)\|\widehat\chi_{N\hat\alpha}\|_q^q.$$

On the other hand, if we use polar coordinates for each component x_j of $x \in \mathbb{C}^{m+1}$, we have

$$\mathcal{I}_q(N) = \prod_{j=0}^{m} \int_{\mathbb{C}} e^{-|x|^2} |x|^{N\alpha_j q} \, d\ell(x)$$
$$= (2\pi)^{m+1} \prod_{j=0}^{m} \int_0^\infty e^{-r^2} r^{N\alpha_j q+1} \, dr = \pi^{m+1} \prod_{j=0}^{m} \Gamma(N\alpha_j q/2 + 1).$$

Hence we obtain

$$\|\widehat{\chi}_{N\hat{\alpha}}\|_q^q = p^m \, \frac{\prod_{j=0}^m \Gamma(Nq\alpha_j/2+1)}{\Gamma(Npq/2+m+1)} \tag{36}$$

and therefore

$$\|\widehat{\chi}_{N\hat{\alpha}}\|^2 = \frac{p^m (N\widehat{\alpha})!}{(Np+m)!}. \tag{37}$$

Since $\widehat{\varphi}^{p\Sigma}_{N\hat{\alpha}} = \widehat{\chi}_{N\hat{\alpha}}/\|\widehat{\chi}_{N\hat{\alpha}}\|$, the identity follows from (36)–(37). □

The following proposition is a direct consequence of Stirling's formula and Proposition 2.2.

Proposition **2.3.** *Let $\alpha \in p\Sigma \cap \mathbb{Z}^m$, and let $J = \{j \in \mathbb{Z} : 0 \le j \le m, \ \alpha_j \neq 0\}$, where $\alpha_0 = p - |\alpha|$. Set $d(\alpha) = 2m - \#J$. Then we have*

$$\|\widehat{\varphi}^{p\Sigma}_{N\alpha}\|_q^{2q} \sim \left(\frac{N}{2\pi}\right)^{(q/2-1)d(\alpha)} \frac{p^{q/2-1}(2\pi)^{r(q-2)}}{(q/2)^{d(\alpha)} \left(\prod_{j\in J}\alpha_j\right)^{q/2-1}};$$

$$\|\widehat{\varphi}^{p\Sigma}_{N\alpha}\|_\infty^2 = p^{-m} \frac{(Np+m)!}{(N\widehat{\alpha})!} \left[\frac{\prod_{j\in J}\alpha_j^{\alpha_j}}{p^p}\right]^N.$$

In particular, again by Stirling's formula, we have

$$\|\widehat{\varphi}^{p\Sigma}_{N\alpha}\|_\infty^2 \sim (2\pi)^r \left(\frac{p}{\prod_{j\in J}\alpha_j}\right)^{1/2} \left(\frac{N}{2\pi}\right)^{d(\alpha)/2}.$$

In the next section, we will obtain similar formulas for monomials on general toric varieties.

3. Pointwise asymptotics on general toric varieties

We now consider the case of general toric varieties. Our first purpose is to find the pointwise asymptotics of the monomials and to prove Theorem 1.2. We then asymptotically determine their $\mathcal{L}^{2k}$-norms.

3.1. **Pointwise asymptotics: interior points.** First of all, we shall consider a sequence α_N of lattice points in NP. We assume that there exists a point $x \in P^o$ such that

$$\alpha_N/N = x + O(N^{-1}). \tag{38}$$

Under the assumption (38), the point α_N/N is in the interior P^o of the polytope P for sufficiently large N. Thus the analysis is performed on the open orbit $(\mathbb{C}^*)^m$, and hence the coordinate

$$z = e^{\rho/2+i\varphi} \in (\mathbb{C}^*)^m, \quad \rho, \varphi \in \mathbb{R}^m$$

will be useful. The moment map μ_P is also invariant under the Hamiltonian $\mathbf{T}^m$-action, and it is well-known ([Fu]) that it induces a diffeomorphism:

$$\bar{\mu}_P : \mathbb{R}^m = (\mathbb{C}^*)^m/\mathbf{T}^m \to P^o, \quad \bar{\mu}_P(\rho) := \mu_P(e^{\rho/2}). \tag{39}$$

In these coordinates, the function b_x^P defined by (7) can be written simply as

$$b_x^P(\rho) = f(x,\rho) - f(x, \rho_x^P), \quad \rho_x^P = \bar{\mu}_P^{-1}(x), \quad f(x,\rho) := \log k(\rho) - \langle\, \rho, x\,\rangle, \tag{40}$$

where the function $k(\rho)$ is the 'polytope character'

$$k(\rho) = \sum_{\beta \in P \cap \mathbb{Z}^m} |c_\beta|^2 e^{\langle\, \rho, \beta\,\rangle}. \tag{41}$$

The vector $\tau_x^P(z)$ in (8) is given by

$$\tau_x^P(z) = \rho - \rho_x^P, \quad z = e^{\rho/2+i\theta}. \tag{42}$$

We also define the symmetric real $m \times m$ matrix

$$A(P,x) := \sum_{\beta \in P} \left| \widehat{m}_\beta^P(e^{\rho_x^P/2}) \right|^2 \beta \otimes \beta - x \otimes x. \tag{43}$$

Lemma **3.1.** *The real symmetric matrix*

$$A(\rho) := \sum_{\beta \in P} |\widehat{m}_\beta^P(e^{\rho/2})|^2 \beta \otimes \beta - \mu_P(e^{\rho/2}) \otimes \mu_P(e^{\rho/2})$$

is positive definite, for all $z \in (\mathbb{C}^*)^m$.

Proof. We must show that $(\lambda \otimes \lambda, A(\rho)) > 0$ for $\lambda \in (\mathbb{R}^m)' \setminus \{0\}$. Consider the vectors $u, v \in \mathbb{R}^{d+1}$ given by $u_\alpha = |\widehat{m}_\alpha^P(e^{\rho/2})|$, $v_\alpha = |\widehat{m}_\alpha^P(e^{\rho/2})|\lambda(\alpha)$. Since $\|u\|^2 = \sum_{\alpha \in P} |\widehat{m}_\alpha^P(e^{\rho/2})|^2 = 1$ and $\mu_P(e^{\rho/2}) = \sum_{\alpha \in P} |\widehat{m}_\alpha^P(e^{\rho/2})|^2 \alpha$, we have

$$\begin{aligned} (\lambda \otimes \lambda, A(\rho)) &= \sum_{\alpha \in P} |\widehat{m}_\alpha^P(e^{\rho/2})|^2 \lambda(\alpha)^2 - \left(\sum_{\alpha \in P} |\widehat{m}_\alpha^P(e^{\rho/2})|^2 \lambda(\alpha) \right)^2 \\ &= \|v\|^2 - \langle u, v\rangle^2 \;=\; \|u\|^2 \|v\|^2 - \langle u, v\rangle^2 \;\geq\; 0\,. \end{aligned} \tag{44}$$

Since $u_\alpha = |\widehat{m}_\alpha^P(e^{\rho/2})| \neq 0$ for all $\alpha \in P \cap \mathbb{Z}^m$ and $v_\alpha/u_\alpha = \lambda(\alpha)$ is not constant on $P \cap \mathbb{Z}^m$, the Cauchy-Schwartz inequality in (44) is strict. □

In particular, the matrix

$$A(P,x) = A(\rho_x^P)$$

is positive definite.

For a point $x \in P^o$, we now define the constant

$$c(P,x) := \frac{1}{\sqrt{\det A(P,x)}}. \tag{45}$$

LEMMA **3.2.** *In the coordinates $z = e^{\rho/2+i\theta}$ on $(\mathbb{C}^*)^m$, the volume form on M_P is given by*

$$\omega_P^m/m! = \frac{1}{(2\pi)^m} \det A(\rho)\, d\rho d\theta\ .$$

Proof. We note that

$$A(\rho) = \sum_{\beta\in P} k_\beta(\rho)\beta\otimes\beta - \left(\sum_{\beta\in P} k_\beta(\rho)\beta\right)\otimes\left(\sum_{\beta\in P} k_\beta(\rho)\beta\right) = \mathrm{Hess}_\rho \log k(\rho), \tag{46}$$

where $k_\beta(\rho) = |\widehat{m}_\beta^P(e^{\rho/2})|^2$ and $k(\rho) = \sum_{\beta\in P}|c_\beta|^2 e^{\langle\,\rho,\beta\,\rangle}$. The conclusion follows from (46), recalling that

$$\omega_P = \Phi_P^{c*}\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\sum_\beta |c_\beta|^2|\chi_\beta(z)|^2 = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log k(\rho). \tag{47}$$

□

It should be noted that $b_x^P(\rho)$ grows as $|\rho|\to\infty$, as stated in the following simple lemma.

LEMMA **3.3.** *Let $K\subset P^o$ be a compact set. Then there exists positive constants $R>0$, $c>0$ such that $f(x,\rho)\geq c$ for $(x,\rho)\in K\times\mathbb{R}^m$, $|\rho|\geq R$.*

Proof. For any $(x,\rho)\in P^o\times\mathbb{R}^m$, we define

$$M(x,\rho) = \max_{\beta\in P\cap\mathbb{Z}^m}\langle\,\rho,\beta-x\,\rangle.$$

If $x\in P^o$, then the polytope $P-x$ contains the origin in its interior. Thus, clearly we have $M(x,\rho)>0$ for any $(x,\rho)\in P^o\times(\mathbb{R}^m\setminus 0)$. Next, we note that the function $(x,\rho)\mapsto M(x,\rho)$ is continuous. To see this, let (x_n,ρ_n) be a sequence such that $(x_n,\rho_n)\to(x,\rho)\in P^o\times\mathbb{R}^m$. Then, for any $\beta\in P\cap\mathbb{Z}^m$,

$$|\langle\,\rho_n,\beta-x_n\,\rangle - \langle\,\rho,\beta-x\,\rangle| \leq C|\rho_n-\rho| + |\rho||x_n-x|.$$

By using this inequality, we can show that

$$|M(x_n,\rho_n)-M(x,\rho)|\leq C|\rho_n-\rho|+|\rho||x_n-x|.$$

Now, for a compact set $K\subset P^o$, we set

$$M(K) = \min_{(x,\rho)\in K\times\mathbb{R}^m, |\rho|=1} M(x,\rho) > 0.$$

We set $c_0 = \min_{\beta\in P\cap\mathbb{Z}^m} |c_\beta|^2$. Since $P\cap\mathbb{Z}^m$ is a finite set, there exists $\beta = \beta(x,\rho)$ such that $M(x,\rho) = \langle\, \rho, \beta(x,\rho) - x\,\rangle$ for (x,ρ) with $x \in P^o$ and $|\rho| = 1$. Thus, for $x \in K \subset P^o$ and $\rho \neq 0$, we have

$$e^{f(x,\rho)} = \sum_\beta |c_\beta|^2 e^{\langle\, \rho,\beta-x\,\rangle} \geq c_0 e^{|\rho|\langle\, \frac{\rho}{|\rho|},\beta(x,\frac{\rho}{|\rho|})-x\,\rangle} = c_0 e^{|\rho| M(x,\frac{\rho}{|\rho|})} \geq c_0 e^{|\rho| M(K)}$$

for $x \in K$ and $\rho \neq 0$, which completes that proof. $\square$

Completion of the proof of Theorem 1.2: Recalling that $z = e^{\rho/2+i\theta}$, we write $|\varphi^P_{\alpha_N}(\rho)|_P$ instead of $|\varphi^P_{\alpha_N}(z)|_P$. By the definition of the Hermitian metric on L_P^N, we have

$$|\varphi^P_{\alpha_N}(\rho)|^2_P = \frac{|\chi^P_{\alpha_N}(\rho)|^2_P}{\|\chi^P_{\alpha_N}\|^2} = \frac{1}{\|\chi^P_{\alpha_N}\|^2}\frac{e^{\langle\, \rho,\alpha_N\,\rangle}}{|\Phi^c_P(e^{\rho/2})|^{2N}},$$

where, in the right hand side, $|\Phi^c_P(e^{\rho/2})|$ denotes the usual norm in $\mathbb{C}^{d+1}$. By the definition (12) of the monomial embedding Φ^c_P, we have

$$|\Phi^c_P(e^{\rho/2})|^2 = k(\rho), \quad \rho \in \mathbb{R}^m.$$

Hence, we have

$$|\varphi^P_{\alpha_N}(\rho)|^2_P = \frac{e^{-Nf(\alpha_N/N,\rho)}}{\|\chi^P_{\alpha_N}\|^2} = \frac{e^{-Nf(x,\rho)}R_N(x,\rho)}{\|\chi^P_{\alpha_N}\|^2}, \quad R_N(x,\rho) = e^{N\langle\, \rho,\alpha_N/N-x\,\rangle},$$

where the function $f(x,\rho)$ for $x \in P^o$ is defined in (40). We note that, since $\alpha_N/N - x = O(N^{-1})$, we have

$$|\partial^L_\rho R_N(x,\rho)| \leq C_L R_N(x,\rho) \tag{48}$$

for every multi-index L. By Lemma 3.2, the $\mathcal{L}^2$-norm of the un-normalized monomial $\chi^P_{\alpha_N}$ is given by

$$\|\chi^P_{\alpha_N}\|^2 = \int_{\mathbb{R}^m} e^{-Nf(\alpha_N/N,\rho)} \det A(\rho)\, d\rho.$$

Here it should be noted that $\det A(\rho)$ is a positive integrable function on $\mathbb{R}^m$. By Lemma 3.3, we can choose $R > 0$, $c > 0$ such that $|\rho^P_x| < R$ and $f(\alpha_N/N,\rho) \geq c$ for any $|\rho| \geq R$ and N. Thus, by choosing a cut-off function $g(\rho)$ suitably, we may write

$$\|\chi^P_{\alpha_N}\|^2 = e^{-Nf(x,\rho^P_x)} \int e^{-Nb^P_x(\rho)} R_N(x,\rho) g(\rho) \det A(\rho)\, d\rho + O(e^{-cN}).$$

Recall that $b^P_x(\rho) = 0$ if and only if $\rho = \rho^P_x$, and that $\rho = \rho^P_x$ is the unique critical point of b^P_x. The Hessian of b^P_x at $\rho = \rho^P_x$ is the positive definite symmetric matrix $A(P,x)$. Thus, by the Morse lemma, there exists a change of coordinates κ from a neighborhood of the origin to a neighborhood of ρ^P_x such that $\kappa(0) = \rho^P_x$ and that

$$b^P_x \circ \kappa(\xi) = \langle\, A(P,x)\xi, \xi\,\rangle/2, \quad |\det D\kappa(0)| = 1.$$

By choosing the cut-off function g suitably, we get

$$\int e^{-Nb^P_x(\rho)} g(\rho) R_N(x,\rho) \det A(\rho)\, d\rho = \int e^{-N\langle\, A(P,x)\xi,\xi\,\rangle} G_N(x,\xi)\, d\xi,$$

where $G_N(x,\xi)$ is a compactly supported function in ξ such that $G_N(x,0) = \det A(P,x)$. By (48), the derivatives of $G_N(x,\xi)$ with respect to ξ are all bounded uniformly in N. Therefore,

by using the Plancherel formula and a formula for the Fourier transform of the Gaussian functions, we obtain

$$\int e^{-Nb_x^P(\rho)} g(\rho) R_N(x,\rho) \det A(\rho)\, d\rho = \left(\frac{N}{2\pi}\right)^{-m/2} \sqrt{\det A(P,x)}(1+O(N^{-1})),$$

which completes the proof. □

Remark: Since $\chi^P_{\alpha_N}$ is a monomial, the asymptotics of $\|\chi^P_{\alpha_N}\|$ is essentially the same calculation as the asymptotics of the L^{2k} norm of another monomial. Thus, determining the pointwise asymptotics of monomials is equivalent to determining the asymptotics of their L^{2k} norms.

3.2. **Pointwise asymptotics: boundary lattice points.** Next, we consider the ray $\mathbb{N}\alpha$ for a lattice point $\alpha \in P \cap \mathbb{Z}^m$, which is allowed to lie in the boundary ∂P. In such a case, we need to work with other coordinates than the usual coordinates on the open orbit $(\mathbb{C}^*)^m$ ([SZ2]), since the open orbit $(\mathbb{C}^*)^m$ does not cover the set $\mu_P^{-1}(\partial P)$.

In the following, we mean that the faces are disjoint, and the facet is a face of codimension one. Thus we call the closed face (or facet) $\bar{F}$ the closure of F in the minimal affine subspace containing F. To describe the coordinates, let v_0 be a vertex of P. Since our polytope P is Delzant, we can choose lattice points $\alpha^1, \ldots, \alpha^m$ in P such that each α^j is in an edge incident to the vertex v_0, and the vectors $v^j := \alpha^j - v_0$ form a basis of $\mathbb{Z}^m$. We choose (open) facets F_j, $j = 1, \ldots, m$ incident at v_0 so that $\alpha^j \notin F_j$.

Lemma **3.4.** *Let $\alpha \in P \cap \mathbb{Z}^m$, and $z \in M_P$. Then, $\chi^P_\alpha(z) = 0$ if and only if*

$$\mu_P(z) \in \bigcup \{\bar{F}\,;\, F \text{ is a facet } \alpha \notin \bar{F}\}. \tag{49}$$

Proof. If $z \in (\mathbb{C}^*)^m$ then automatically we have $\chi^P_\alpha(z) \neq 0$. Thus, we may assume that $z \in \mu_P^{-1}(\partial P)$. First, assume that $\mu_P(z) \in \bar{F}$ for some closed facet $\bar{F}$ which does not contain α. The last formula in (23) for the moment map is globally defined, for the function $|\widehat{m}^P_\beta|^2$ is globally defined. Since $\mu_P(z) \in \bar{F}$, the coefficients in $\mu_P(z)$ of the lattice points $\beta \notin \bar{F}$ must vanish. Thus, we have $\chi^P_\alpha(z) = 0$. Conversely, assume that $\mu_P(z) \in \partial P$ is not in the set described in (49). In our convention, the faces are disjoint and the boundary ∂P is the disjoint union of faces. Thus, that $\mu_P(z) \in \partial P$ is not in the set described in (49) is equivalent to say that there exists an open face E such that $\alpha \in \bar{E}$ and $\mu_P(z) \in E$. Let $v_1, \ldots, v_l$ be the set of vertex of $\bar{E}$. Since v_j and α are lattice points, there exists a positive integer n_0 such that $n_0\alpha = \sum_{j=1}^l n_j v_j$ with n_j integer such that $\sum n_j = n_0$. Thus we have

$$(\chi^P_\alpha)^{\otimes n_0}(z) = (\chi^P_{v_1})^{\otimes n_1}(z) \otimes \cdots \otimes (\chi^P_{v_l})^{\otimes n_l}(z).$$

Since $\mu_P(z)$ is in the interior E of the face (polytope) $\bar{E}$, each $\chi^P_{v_j}(z)$ can not vanish, and hence $\chi^P_\alpha(z) \neq 0$. □

Using Lemma 3.4, we set

$$U_{v_0} := \{z \in M_P\,;\, \chi^P_{v_0}(z) \neq 0\},$$

which covers M_P as v_0 varies over all vertices. We define

$$\eta : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m, \quad \eta(z) = \eta_j(z) := (z^{v^1}, \ldots, z^{v^m}). \tag{50}$$

The map η is a diffeomorphism and the inverse is given by

$$z : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m, \quad z(\eta) = (\eta^{\Gamma e^1}, \ldots, \eta^{\Gamma e^m}),$$

where e^j is the standard basis for $\mathbb{R}^m$ (or for $\mathbb{C}^m$ over $\mathbb{C}$), and Γ is an $m \times m$-matrix with $\det \Gamma = \pm 1$ and integer coefficients defined by

$$\Gamma v^j = e^j, \quad v^j = \alpha^j - v_0.$$

By definition, we have the obvious formula:

$$\chi^P_{\alpha^j}(z) = \eta_j(z)\chi^P_{v^0}(z), \quad z \in (\mathbb{C}^*)^m.$$

By Lemma 3.4, $\eta_j(z) \to 0$ if $z \in U_{v_0}$, $z \to \mu_P^{-1}(\bar{F}_j)$. Since $\alpha^j \notin \bar{F}_j$, we have

$$(\chi^P_{\alpha^j})^{-1}(0) \cap U_{v_0} = \mu_P^{-1}(\bar{F}_j).$$

The set $U_{v_0} \setminus (\mathbb{C}^*)^m$ is the union of the sets $\mu_P^{-1}(\bar{F}_j)$, and hence the map η extends a homeomorphism:

$$\eta : U_{v_0} \to \mathbb{C}^m, \quad \eta(z_0) = 0, \quad z_0 = \text{ the fixed point corresponding to } v_0.$$

By this homeomorphism, the set $\mu_P^{-1}(\bar{F}_j)$ corresponds to the set $\{\eta \in \mathbb{C}^m \,;\, \eta_j = 0\}$. This coordinate $\eta = (\eta_1, \ldots, \eta_m)$ is useful to explain toric subvarieties corresponding to faces. Namely, let $\bar{F}$ be a closed face with $\dim F = m - r$ which contains v_0. Since $v_0 \in \bar{F}$, we can choose $F_{i_1}, \ldots, F_{i_r}$ such that $\bar{F} = \bar{F}_{i_1} \cap \cdots \cap \bar{F}_{i_r}$. Then the subvariety $\mu_P^{-1}(\bar{F})$ corresponding $\bar{F}$ is expressed, in the coordinate neighborhood U_{v_0}, by

$$\mu_P^{-1}(\bar{F}) \cap U_{v_0} = \{\eta \in \mathbb{C}^m \,;\, \eta_{i_j} = 0, \quad j = 1, \ldots, r\}.$$

Now, we fix a lattice point α in a (relatively open) face F of dimension $\dim F = m - r$ such that $v_0 \in \bar{F}$. Without loss of generality, we may assume that $\bar{F} = \bar{F}_1 \cap \cdots \cap \bar{F}_r$.

To state a result for the lattice point α in the boundary corresponding to Theorem 1.2, we need to find a function corresponding to the function b^P_α. Since our coordinate η is based on the lattice points $\alpha^j - v_0$, it is reasonable to introduce a new polytope defined by the affine linear transformation

$$\tilde{\Gamma} : \mathbb{R}^m \ni u \to \Gamma u - \Gamma v_0 \in \mathbb{R}^m,$$

which maps $\mathbb{Z}^m$ bijectively onto itself. We set $Q := \tilde{\Gamma}(P)$. Then Q is contained in the positive orthant $\{x \in \mathbb{R}^m \,;\, x_j \geq 0\}$, and we have $\tilde{\Gamma}(\bar{F}_j) = \{x \in Q \,;\, x_j = 0\}$. The face of Q corresponding to F is then given by

$$Q_F := \tilde{\Gamma}(\bar{F}) = \{x \in Q \,;\, x_j = 0, \quad j = 1, \ldots, r\}.$$

We denote a point in $U_{v_0} \cong \mathbb{C}^m$ as $\eta = (\xi, \zeta) \in \mathbb{C}^m = \mathbb{C}^r \times \mathbb{C}^{m-r}$. In this expression, $\zeta = (0, \zeta)$ is a local coordinate of the submanifold $\mu_P^{-1}(\bar{F})$. The modulus square of the monomials $|\chi^P_{N\alpha}|^2_P$ with $\alpha \in P \cap \mathbb{Z}^m$ is, in this coordinate, given by

$$\begin{aligned} |\chi^P_{N\alpha}(z)|^2 &= \frac{|\eta^{\tilde{\Gamma}(\alpha)}|^{2N}}{K(\eta)^N}, \\ K(\eta) &= \sum_{\gamma \in Q \cap \mathbb{Z}^m} a_\gamma |\eta^\gamma|^2, \quad a_\gamma = |c_{\tilde{\Gamma}^{-1}(\gamma)}|^2. \end{aligned} \tag{51}$$

We then introduce the 'moment map' corresponding to the face F by:

$$\mu_F : \mathbb{R}^{m-r} \to Q_F, \quad \mu_F(\rho) = \sum_{(0,\nu)\in Q_F} \frac{a_\nu e^{\langle \rho,\nu \rangle}}{k_F(\rho)}(0,\nu), \tag{52}$$

where $a_\nu = |c_{\tilde{\Gamma}^{-1}(0,\mu)}|^2$, and the function $k_F(\rho)$ is given by

$$k_F(\rho) := \sum_{(0,\nu)\in Q_F} a_\nu e^{\langle \rho,\nu \rangle}. \tag{53}$$

As mentioned above, the submanifold $\mu_P^{-1}(\bar{F})$ has the coordinate $\zeta \mapsto (0,\zeta) \in \mathbb{C}^r \times \mathbb{C}^{m-r}$, and the torus $\mathbf{T}^{m-r}$ acts on it. Thus, it is natural to use the coordinate

$$\zeta = e^{\rho/2+i\theta}, \quad \rho,\theta \in \mathbb{R}^{m-r}.$$

Then, we write $\eta = (\xi,\zeta) = (\xi,\rho)$ for $\zeta = e^{\rho/2}$. Since we have assumed $\alpha \in F$, we may write

$$\tilde{\Gamma}(\alpha) = (0,\tilde{\alpha}) \in Q_F.$$

We also define

$$\begin{aligned} s_\alpha(\xi,\rho) &:= \log K(\xi,\rho) - \langle \rho,\tilde{\alpha} \rangle = \log(k_F(\rho) + \ell_F(\xi,\rho)) - \langle \rho,\tilde{\alpha} \rangle, \\ \ell_F(\xi,\rho) &= \sum_{(\mu,\nu)\in Q,\mu\neq 0} a_{(\mu,\nu)} |\xi^\mu|^2 e^{\langle \rho,\nu \rangle}. \end{aligned} \tag{54}$$

Proposition **3.5.** *In the coordinate $\eta = (\xi,\zeta)$ as above, we write $|\varphi^P_{N\alpha}(\xi,\rho)|_P$ for the modulus square of the monomial $|\varphi^P_{N\alpha}|^2_P$. We also write $\tilde{\Gamma}(\alpha) = (0,\tilde{\alpha})$, which is in the interior of the polytope Q_F. Then we have*

$$|\varphi^P_{N\alpha}(\xi,\rho)|^2_P = (2\pi)^r \left(\frac{N}{2\pi}\right)^{(m+r)/2} \frac{e^{-N\Psi_\alpha(\xi,\rho)}}{\sqrt{\det A(F,\alpha)}}(1+O(N^{-1})),$$

where the function $\Psi_\alpha(\xi,\rho)$ is given by

$$\Psi_\alpha(\xi,\rho) = s_\alpha(\xi,\rho) - s_\alpha(0,\rho^F_\alpha) \quad \rho^F_\alpha = \mu_F^{-1}(\tilde{\alpha}) \in \mathbb{R}^{m-r}. \tag{55}$$

and $(m-r)\times(m-r)$ positive definite matrix $A(F,\alpha)$ is given by

$$A(F,\alpha) = \sum_{(0,\nu)\in Q_F} \frac{a_\nu e^{\langle \rho^F_\alpha,\nu \rangle}}{k_F(\rho^F_\alpha)} \nu\otimes\nu - \tilde{\alpha}\otimes\tilde{\alpha}. \tag{56}$$

We shall prove Proposition 3.5 in the rest of this subsection. First of all, we need the following simple lemma:

Lemma **3.6.** *In the coordinate $(\xi,\zeta = e^{\rho/2+i\theta})$ on $\mathbb{C}^r \times (\mathbb{C}^*)^{m-r} \subset U_{v_0}$, the volume form $\omega_P^m/m!$ is given by*

$$\frac{\omega_P^m}{m!} = \frac{1}{\pi^r(2\pi)^{m-r}} L(\xi,\rho)\, dm(\xi) d\rho d\theta, \tag{57}$$

where $dm(\xi)$ denotes the Lebesgue measure on $\mathbb{C}^r$, and the function $L(\xi,\rho)$ is given by the determinant of the following $m\times m$ matrix:

$$L(\xi,\rho) = \det \begin{pmatrix} \frac{\partial^2 \log K}{\partial\xi\partial\bar{\xi}} & \frac{\partial^2 \log K}{\partial\xi\partial\rho} \\ \frac{\partial^2 \log K}{\partial\rho\partial\bar{\xi}} & \frac{\partial^2 \log K}{\partial\rho\partial\rho} \end{pmatrix}, \tag{58}$$

where $K(\xi,\rho)$ is the function defined in (51).

Proof. For the function $f(\xi,\zeta)$ on $\mathbb{C}^r\times(\mathbb{C}^*)^{m-r}$ independent of the variable θ in $\zeta=e^{\rho/2+i\theta}$, then the derivatives $(\partial f)/(\partial\zeta_j)$, $(\partial f)/(\partial\bar{\zeta}_j)$ is given, respectively, by

$$\frac{\partial f}{\partial\zeta_j}=\frac{1}{\zeta_j}\frac{\partial f}{\partial\rho_j},\quad \frac{\partial f}{\partial\bar{\zeta}_j}=\frac{1}{\bar{\zeta}_j}\frac{\partial f}{\partial\rho_j}.$$

We write $\eta=(\xi,\zeta)$. Then, by this relation, we have

$$\det\left(\frac{\partial^2\log K}{\partial\eta_j\partial\bar{\eta}_k}\right)=\left(\prod_{j=1}^{m-r}|\zeta_j|^2\right)^{-1}L(\xi,\rho),$$

where $L(\xi,\rho)$ is given by (58). But, we have

$$dm(\zeta)=\frac{1}{2^{m-r}}\left(\prod_{j=1}^{m-r}|\zeta_j|^2\right)d\rho d\theta,$$

where $dm(\zeta)$ denotes the Lebesgue measure on $\mathbb{C}^{m-r}$. Combining this with (47), we obtain the assertion. □

The following lemma can be shown by the same argument as in the proof of Lemma 3.1, and we shall omit the proof.

LEMMA **3.7.** *The $(m-r)\times(m-r)$ matrix defined by*

$$A_F(\rho):=\partial\mu_F(\rho)=\sum_{(0,\nu)\in Q_F}\frac{a_\nu e^{\langle\rho,\nu\rangle}}{k_F(\rho)}\nu\otimes\nu-\mu_F(\rho)\otimes\mu_F(\rho)$$

is positive definite for every $\rho\in\mathbb{R}^{m-r}$.

Note that, the map $\mu_F:\mathbb{R}^{m-r}\to Q_F^o$ is a diffeomorphism, and the lattice point $\tilde{\alpha}$ is in Q_F^o. Thus, the vector $\rho_\alpha^F=\mu_F^{-1}(\tilde{\alpha})$ is well-defined. Hence, the $(m-r)\times(m-r)$ matrix

$$A(F,\alpha)=A_F(\rho_\alpha^F)$$

is positive definite.

Completion of proof of Proposition 3.5. The modulus square of the monomial $|\varphi_{N\alpha}^P|_P^2$ in this coordinate is given by

$$\begin{aligned}|\varphi_{N\alpha}^P(\xi,\rho)|_P^2&=\frac{|\chi_{N\alpha}^P(\xi,\rho)|_P^2}{\|\chi_{N\alpha}^P\|^2}\\&=\frac{1}{\|\chi_{N\alpha}^P\|^2}e^{-Ns_\alpha(\xi,\rho)},\end{aligned}$$

where the function $s_\alpha(\xi,\rho)$ is defined by (54). Thus, as in the proof of Theorem 1.2, what we need to analyze is the $\mathcal{L}^2$-norm

$$\|\chi_{N\alpha}^P\|^2=\frac{1}{\pi^r}\int_{\mathbb{C}^r\times\mathbb{R}^{m-r}}e^{-Ns_\alpha(\xi,\rho)}L(\xi,\rho)\,dm(\xi)d\rho,\tag{59}$$

where we have used Lemma 3.6. We note that the function $\ell_F(\xi,\rho)$ defined in (54) is of the form

$$\begin{gathered} \ell_F(\xi,\rho)=\sum_{k=1}^r f_k(\rho)|\xi_k|^2+r(\xi,\rho), \\ f_k(\rho)=\sum_{(e_k^r,\nu)\in Q} a_{(e_k^r,\nu)}e^{\langle\rho,\nu\rangle}, \quad r(\xi,\rho)=\sum_{(\mu,\nu)\in Q,|\mu|\geq 2} a_{(\mu,\nu)}|\xi^\mu|^2 e^{\langle\rho,\nu\rangle}, \end{gathered} \tag{60}$$

where e_k^r are the standard basis for $\mathbb{R}^r$. The function $r(\xi,\rho)$ is of order ≥ 4 in ξ, and hence its derivative up to the second order vanish at $\xi=0$. Thus, the function $s_\alpha(\xi,\rho)$ has only one critical point $(\xi,\rho)=(0,\rho_\alpha^F)$. It is not hard to show that the Hessian $Hs_\alpha(0,\rho_\alpha)$ of the function (54) at the critical point $(0,\rho_\alpha^F)$ is given by

$$Hs_\alpha(0,\rho_\alpha^F)=\begin{pmatrix} \frac{2f_1(\rho_\alpha^F)}{k_F(\rho_\alpha^F)} & & & & & & \\ & \ddots & & & & & \\ & & \frac{2f_r(\rho_\alpha^F)}{k_F(\rho_\alpha^F)} & & & & \\ & & & \frac{2f_1(\rho_\alpha^F)}{k_F(\rho_\alpha^F)} & & & \\ & & & & \ddots & & \\ & & & & & \frac{2f_r(\rho_\alpha^F)}{k_F(\rho_\alpha^F)} & \\ & & & & & & A(F,\alpha) \end{pmatrix}, \tag{61}$$

where the $(m-r)\times(m-r)$-matrix $A(F,\alpha)$ is given by (56), and we have used the coordinate (x_j,y_j,ρ) with $\xi_j=x_j+iy_j$. Here, it should be noted that the polytope $Q=\tilde{\Gamma}(P)$ contains the standard basis in $\mathbb{R}^m=\mathbb{R}^r\times\mathbb{R}^{m-r}$. Thus the lattice points $(e_j^r,0)$ with the standard basis e_j^r in $\mathbb{R}^r$ is in the polytope Q, and hence the functions f_j are all positive. This combined with Lemma 3.7 shows that the Hessian $Hs_\alpha(0,\rho_\alpha)$ is positive definite. By the same argument as in the proof of Lemma 3.3, the function $k_F(\rho)+\ell_F(\xi,\rho)$ tends to ∞ as $|\xi|+|\rho|\to\infty$. Therefore, by the standard Laplace method as in the proof of Theorem 1.2, we have

$$\|\chi_{N\alpha}^P\|^2=\frac{1}{\pi^r}\left(\frac{N}{2\pi}\right)^{-(m+r)/2}\frac{e^{-Ns_\alpha(0,\rho_\alpha^F)}}{\sqrt{\det Hs_\alpha(0,\rho_\alpha^F)}}L(0,\rho_\alpha^F)(1+O(N^{-1})). \tag{62}$$

A direct computation will show that

$$\det Hs_\alpha(0,\rho_\alpha^F)=\frac{\det A(F,\alpha)}{k_F(\rho_\alpha^F)^{2r}}\left(\prod_{j=1}^r 2f_j(\rho_\alpha^F)\right)^2, \quad L(0,\rho)=\frac{\det A_F(\rho)}{k_F(\rho_\alpha^F)^r}\left(\prod_{j=1}^r f_j(\rho)\right), \tag{63}$$

and hence, the asymptotics (62) can be written in the form:

$$\|\chi_{N\alpha}^P\|^2=\frac{1}{(2\pi)^r}\left(\frac{N}{2\pi}\right)^{-(m+r)/2}\sqrt{\det A(F,\alpha)}e^{-Ns_\alpha(0,\rho_\alpha^F)}(1+O(N^{-1})).$$

Dividing $|\chi_{N\alpha}|_P^2$ by the above, we conclude the assertion. $\square$

When our fixed lattice point α is a vertex, say $\alpha=v_0$ in the description of the coordinate η, the matrix $A(F,\alpha)$ is not defined suitably. However, clearly the similar asymptotics can be deduce by the same method.

Proposition 3.8. *Suppose that α is a vertex of the polytope P. Then we have*

$$|\varphi^P_{N\alpha}(\eta)|^2_P = \left(\frac{N}{|c_\alpha|^2}\right)^m e^{-N(\log K(\eta) - \log|c_\alpha|^2)}(1+O(N^{-1})),$$

where the function $K(\eta)$ on $\mathbb{C}^m$ is given by (51). *We also have $|c_\alpha|^2 = |\chi^P_\alpha(z_\alpha)|^{-2}$ where z_α is the fixed point for the Hamiltonian $\mathbf{T}^m$-action such that $\mu_P(z_\alpha) = \alpha$.*

Proof. In the description of the coordinate η, we put $\alpha = v_0$. Then, clearly we have $K(0) = |c_\alpha|^2 = |\chi^P_\alpha(z_\alpha)|^{-2}$, where the fixed point z_α corresponds to the origin $\eta = 0$ and the function $K(\eta)$ is defined in (51). In this case, we just use the coordinate η itself without change of variable. In this coordinate, the monomial $|\chi^P_{N\alpha}|^2_P$ is given by

$$|\chi^P_{N\alpha}(\eta)|^2_P = e^{-N\log K(\eta)}.$$

The volume measure $\omega^m_P/m!$ is of the form:

$$\frac{\omega^m_P}{m!} = \frac{1}{\pi^m}\det L(\eta) dm(\eta), \quad L(\eta) = \left(\frac{\partial^2 \log K(\eta)}{\partial\eta_j\partial\bar{\eta}_k}\right) dm(\eta)$$

with the Lebesgue measure $dm(\eta)$ on $\mathbb{C}^m$. In is straight forward to see that the critical point of the function $\log K(\eta)$ is the origin, and the determinant of the Hessian at the origin is given by

$$H(\log K)(0) = \begin{pmatrix} \frac{2L(0)}{K(0)} & 0 \\ 0 & \frac{2L(0)}{K(0)} \end{pmatrix}.$$

By using these facts with the Laplace method, we obtain the assertion. □

3.3. **Moments and $\mathcal{L}^{2k}$ norms.** As an application of the pointwise estimates, one can prove that eigenfunctions 'localize on tori'. We also determine $\mathcal{L}^{2k}$ norm of the monomials. The following is easily shown by using Theorem 1.2, Propositions 3.5 and 3.8, and the argument is the same as in their proofs, and hence we shall omit the proof (see also the proof of Theorem 3.11).

Proposition 3.9. (i) *Let $\alpha_N \in NP \cap \mathbb{Z}$ be a sequence of lattice points satisfying* (38) *for some point $x \in P^o$. Then, the measure $|\varphi^P_{\alpha_N}|^2_P d\mathrm{Vol}_{M_P}$ weak*-converges to the normalized Haar measure on the m-dimensional torus $\mu_P^{-1}(x)$, i.e.,*

$$\int_{M_P} \sigma|\varphi^P_{\alpha_N}|^2_P d\mathrm{Vol}_{M_P} \to \int_{\mu_P^{-1}(x)} \sigma d\theta$$

for $\sigma \in \mathcal{C}(M_P)$.

(ii) *Let α be a lattice point in P with* $\dim \mu_P^{-1}(\alpha) = m - r$. *Then, we have*

$$\text{w}^*\text{-}\lim_{N\to\infty} |\varphi^P_{N\alpha}|^2_P = d\theta_{\mu_P^{-1}(\alpha)},$$

where $d\theta_{\mu_P^{-1}(\alpha)}$ denotes the normalized Haar measure on the $(m-r)$-dimensional torus $\mu_P^{-1}(\alpha) \cong \mathbf{T}^{m-r}$.

In the above proposition, if α is a vertex, then the left hand side denotes the Dirac measure at the fixed point $z_\alpha \in M_P$ of the Hamiltonian $\mathbf{T}^m$-action corresponding to α.

We note that $|\widehat{\varphi}^P_{N\alpha}|$ is invariant under the $\mathbf{T}^m$ action. We denote the Hilbert space of $\mathbf{T}^m$ invariant functions by $\mathcal{L}^2_{\mathrm{inv}}(M_P)$. We can restate the conclusion as follows: if $\sigma \in C^\infty_{\mathrm{inv}}(M_P)$,

then we can regard it as a function on the polytope P. We can also regard $|\widehat{\varphi}^P_{N\alpha}|^2$ as a function, say $|\widetilde{\varphi}^P_\alpha(I)|^2$, on P, equipped with action variables I. We then have:

Corollary **3.10.** *For any $\alpha \in P \cap \mathbb{Z}^m$, we have $|\widetilde{\varphi}^P_{N\alpha}|^2 dI \to \delta_\alpha$; i.e., $\int_P \sigma |\widetilde{\varphi}^P_\alpha|^2\, dI \to \sigma(\alpha)$, for $\sigma \in \mathcal{C}(P)$. For $\alpha_N \in NP \cap \mathbb{Z}^m$ satisfying (38) with a point $x \in P^o$, we have $|\widetilde{\varphi}^P_{\alpha_N}|^2 dI \to \delta_x$.*

Next, we determine the asymptotics of the $\mathcal{L}^{2k}$-norm of the $\mathcal{L}^2$-normalized monomials.

Theorem **3.11.** (i) *Let $\alpha_N \in NP \cap \mathbb{Z}^m$ be a sequence of lattice points satisfying (38) with a point $x \in P^o$. Let $\|\varphi^P_\gamma\|_{2k}$ denote the $\mathcal{L}^{2k}$-norm of the $\mathcal{L}^2$-normalized monomial φ^P_γ with the weight $\gamma \in NP \cap \mathbb{Z}^m$. Then we have*

$$\|\varphi^P_{\alpha_N}\|^{2k}_{2k} = \frac{1}{k^{m/2}} \left(\frac{N}{2\pi}\right)^{m/2} \frac{1}{(\det A(P,x))^{(k-1)/2}} (1 + O_k(N^{-1})), \tag{64}$$

where the $O_k(N^{-1})$ depends on k.

(ii) *Let $\alpha \in P$ be a lattice point with $\dim \mu_P^{-1}(\alpha) = m - r$. Then, for $r \le m-1$, we have*

$$\|\widehat{\varphi}^P_{N\alpha}\|^{2k}_{2k} = \frac{1}{k^{(m+r)/2}} \left(\frac{N}{2\pi}\right)^{(k-1)(m+r)/2} \left(\frac{(2\pi)^r}{\sqrt{\det A(F,\alpha)}}\right)^{k-1} (1 + O_k(N^{-1})). \tag{65}$$

For a vertex α, we have

$$\|\widehat{\varphi}^P_{N\alpha}\|^{2k}_{2k} = \frac{1}{k^m} \left(\frac{N}{|c_\alpha|^2}\right)^{(k-1)m} (1 + O_k(N^{-1})). \tag{66}$$

Proof. The proof of (64) and (66) is the same as that for (65), so we shall give a proof of (65) only. By Theorem 1.2 and Proposition 3.5, we have

$$\begin{aligned}&\|\widehat{\varphi}^P_{N\alpha}\|^{2k}_{2k} \\ &= \frac{(2\pi)^{kr}}{\pi^r} \left(\frac{N}{2\pi}\right)^{k(m+r)/2} \frac{1}{(\det A(F,\alpha))^{k/2}} \int_{\mathbb{C}^r \times \mathbb{R}^{m-r}} e^{-Nk\Psi_\alpha(\xi,\rho)} L(\xi,\rho)\, dm(\xi) d\rho (1 + O_k(N^{-1})),\end{aligned}$$

where Ψ_α is given by (55). As in the proof of Proposition 3.5, the critical point of Ψ_α is only the point $(0, \rho^F_\alpha)$ with $\rho^F_\alpha = \mu_F^{-1}(\tilde{\alpha})$, $\tilde{\Gamma}(\alpha) = (0, \tilde{\alpha})$, and which is non-degenerate. We have $\Psi_\alpha(0, \rho^F_\alpha) = 0$. Thus, by the standard Laplace method, we have

$$\|\widehat{\varphi}^P_{N\alpha}\|^{2k}_{2k} = \frac{(2\pi)^{kr}}{\pi^r} \left(\frac{N}{2\pi}\right)^{(k-1)(m+r)/2} \frac{1}{(\det A(F,\alpha))^{k/2}} \frac{L(0,\rho_\alpha)}{\det H\Psi_\alpha(0,\rho_\alpha)} (1 + O_k(N^{-1})).$$

Therefore, the assertion follows from (63) in the proof of Proposition 3.5. □

By using Proposition 3.5, we can determine the limit of the sup-norm $\|\varphi^P_{N\alpha}\|^2_\infty$.

Proposition **3.12.** (i) *Let $\alpha_N \in NP \cap \mathbb{Z}^m$ satisfy (38) with a point $x \in P^o$. Then we have*

$$\lim_{N\to\infty} \left(\frac{N}{2\pi}\right)^{m/2} \|\varphi^P_{\alpha_N}\|^2_\infty = \frac{1}{\sqrt{\det A(P,x)}}. \tag{67}$$

(ii) *Let α be a lattice point with* $\dim \mu_P^{-1}(\alpha) = m - r$, $r \leq m-1$. *Let* (ξ_N, ρ_N) *be the point where* $|\varphi^P_{N\alpha}|^2_P$ *attains its maximum. Then, we have*

$$\lim_{N\to\infty} \left(\frac{N}{2\pi}\right)^{-(m+r)/2} \|\varphi^P_{N\alpha}\|^2_\infty = \frac{(2\pi)^r}{\sqrt{\det A(F,\alpha)}}. \tag{68}$$

For a vertex $\alpha \in P$, we have

$$\lim_{N\to\infty} N^{-m}\|\varphi^P_{N\alpha}\|^2_\infty = |c_\alpha|^{-2m} = |\chi^P_\alpha(z_\alpha)|^{2m}_P, \tag{69}$$

where $z_\alpha \in M_P$ is the unique fixed point for the $\mathbf{T}^m$ such that $\mu_P(z_\alpha) = \alpha$.

Proof. We only give a proof of (68). The function $\Psi_\alpha(\xi,\rho)$ defined in (55) attains its minimum at the point $(0,\rho^F_\alpha)$, and it tends to ∞ as $|\xi|+|\rho| \to \infty$. Thus, the function $(\frac{N}{2\pi})^{-(m+r)/2}|\varphi^P_{N\alpha}|^2_P$ is of order $O(e^{-cN})$ outside a compact neighborhood of $(0,\rho^F_\alpha)$. On a compact neighborhood B of $(0,\rho^F_\alpha)$, we have

$$|\varphi^P_{N\alpha}(0,\rho^F_\alpha)|^2_P \leq \sup_{(\xi,\rho)\in B} |\varphi^P_{N\alpha}(\xi,\rho)|^2_P \leq \left(\frac{N}{2\pi}\right)^{(m+r)/2} \frac{(2\pi)^r}{\sqrt{\det A(F,\alpha)}}(1+O(N^{-1})).$$

Now (68) follows from the above inequality. The same argument with Proposition 3.5 shows (69). □

3.4. **Asymptotics on projective space.** The values of the $\mathcal{L}^{2k}$ norm in Proposition 2.2 and in the projective-space case of Theorem 1.6 may seem to be different. However, we can check that these two coincide by noting the following simple lemma.

Lemma **3.13.** *The determinant* $\det A(p\Sigma,\alpha)$ *of the matrix* $A(p\Sigma,\alpha)$ *is given by*

$$\det A(p\Sigma,\alpha) = \frac{(p-|\alpha|)\alpha_1\cdots\alpha_m}{p}.$$

Proof. By applying the differential operator $x_j\partial_{x_j}$ twice to the formula $(1+\sum_{l=1}^m x_l)^p = \sum_{|\beta|\leq p}\binom{p}{\beta}x^\beta$ for a vector $x \in \mathbb{R}^m$, we have

$$p(p-1)x_ix_j(1+\sum_l x_l)^{p-2} + px_j(1+\sum_l x_l)^{p-1}\delta_{ij} = \sum_{|\beta|\leq p}\binom{p}{\beta}x^\beta\beta_i\beta_j,$$

where δ_{ij} is the Kronecker's delta, and we have set $\binom{p}{\beta} = p!/\beta!(p-|\beta|)!$. We put $r = \sum e^{(\rho_\alpha)_j}$. Then, by (35), we have $e^{\rho_\alpha} = \frac{1+r}{p}\alpha$ and $|\widehat{m}^{p\Sigma}_\beta(e^{\rho_\alpha/2})|^2 = e^{\langle \rho_\alpha,\beta\rangle}\binom{p}{\beta}/(1+r)^p$. From this the second equation follows. Substituting $\frac{1+r}{p}\alpha$ for x in the above formula, we obtain

$$A(p\Sigma,\alpha)_{i,j} = \sum_{|\beta|\leq p}\binom{p}{\beta}\frac{e^{\langle\rho,\beta\rangle}}{(1+r)^p}\beta_i\beta_j - \alpha_i\alpha_j = \alpha_j\delta_{ij} - \frac{1}{p}\alpha_i\alpha_j.$$

We set $D_m(\alpha_1,\ldots,\alpha_m) = \det A(p,\alpha)$ with $\alpha = (\alpha_1,\ldots,\alpha_m)$. Then a simple computation shows that

$$\frac{D_m(\alpha_1,\ldots,\alpha_m)}{\alpha_1\cdots\alpha_m} = \frac{D_{m-1}(\alpha_2,\ldots,\alpha_m)}{\alpha_2\cdots\alpha_m} - \frac{1}{p}\alpha_1.$$

Thus the lemma follows by induction on the dimension m. □

Note that, for the simplex $p\Sigma$, the faces containing the origin is again a simplex in lower dimensional vector space. Therefore, Lemma 3.13 can be applied to compute $\det A(F,\alpha)$.

In the case of projective monomials, the function $b_\alpha^{p\Sigma}$ can be expressed as follows.

Proposition **3.14.** *For $z \in (\mathbb{C}^*)^m$,*

$$b_\alpha^{p\Sigma}(z) = p\log(1+|z|^2) - \log|z^\alpha|^2 + \langle\, \widehat{\alpha}, \log\widehat{\alpha}\,\rangle - p\log p,$$

where $\log\widehat{\alpha} = (\log\widehat{\alpha}_0, \dots, \log\widehat{\alpha}_m)$.

This is a direct consequence of the formulas $e^{\rho_\alpha} = \frac{1+r}{p}\alpha$, $r = \sum e^{(\rho_\alpha)_j}$ and $(1+\sum_l x_l)^p = \sum_{|\beta|\le p}\binom{p}{\beta}x^\beta$ as mentioned before.

The pointwise asymptotics of the $\mathcal{L}^2$-normalized projective monomials is given in the following proposition.

Proposition **3.15.** (1) *For $z \in (\mathbb{C}^*)^m$,*

$$|\widehat{\varphi}_{N\alpha}^{p\Sigma}(z)|^2 = \left(\frac{N}{2\pi}\right)^{m/2} \frac{p^{1/2}e^{-Nb_\alpha^{p\Sigma}(z)}}{\sqrt{(p-|\alpha|)\alpha_1\cdots\alpha_m}}(1+O(N^{-1})),$$

uniformly $(\mathbb{C}^*)^m$.

(2) *For $z = e^{(\rho_\alpha + u/\sqrt{N})/2 + i\theta}$, we have*

$$|\widehat{\varphi}_{N\alpha}^{p\Sigma}(z)|^2 = \left(\frac{N}{2\pi}\right)^{m/2} \frac{p^{1/2}e^{-(\langle\,\Delta(\alpha)u,u\,\rangle - \frac{1}{p}\langle\,\alpha,u\,\rangle^2)/2}}{\sqrt{(p-|\alpha|)\alpha_1\cdots\alpha_m}}(1+O(N^{-1/2}))$$

uniformly for $|u|\le c$, where $\Delta(\alpha)$ is the diagonal matrix with entries $\alpha_1,\dots,\alpha_m$.

The assertion (1) in the above is a restatement of Theorem 1.2 for the projective monomials. The assertion (2) follows from (1) and a Taylor expansion of the function $b_\alpha^{p\Sigma}$.

4. Asymptotics of distribution functions

In this section, we find asymptotics of rescaled and un-rescaled distribution functions. Fix a lattice point α in P, and let r denote the codimension of the face of P (possibly the open face P^o) containing α. In analogy with (45), we define the constant $c(P,\alpha)$ by

$$c(P,\alpha) := \begin{cases} \dfrac{(2\pi)^r}{\sqrt{\det A(F,\alpha)}} & \text{if } r < m \\ \dfrac{(2\pi)^m}{|c_\alpha|^{2m}} = (2\pi|\chi_{N\alpha}^P(z_\alpha)|_P^2)^m & \text{if } \alpha \text{ is a vertex.} \end{cases} \tag{70}$$

In the following discussion, we give the details for the case where $\alpha_N = N\alpha$ with a lattice point $\alpha \in P\cap\mathbb{Z}^m$. For general $\alpha_N \in NP\cap\mathbb{Z}^m$ satisfying (38) for a point $x \in P^o$, one needs only to put $r=0$ and replace $N\alpha$ and α by α_N and x.

4.1. **Rescaled distribution functions.** We would like to understand the limit distributions of the measures $|\widehat{\varphi}^P_{N\alpha}|^2_* d\mathrm{Vol}_{M_P}$, namely, the limit of its distribution function

$$D_{N\alpha}(t) := \mathrm{Vol}_{M_P}\{z\,;\,|\widehat{\varphi}^P_{N\alpha}(z)|^2 > t\}. \tag{71}$$

However, by Theorem 1.6, the k-th moments of the measure $|\widehat{\varphi}^P_{N\alpha}|^2_* d\mathrm{Vol}_{M_P}$ tends to infinity as N tends to infinity. Therefore, we need to re-normalize the monomials.

We write

$$\mathrm{dv}^r_N = \left(\frac{N}{2\pi}\right)^{(m+r)/2} d\mathrm{Vol}_{M_P}, \quad f_{N\alpha} = \left(\frac{N}{2\pi}\right)^{-(m+r)/4} \varphi^P_{N\alpha}(z) \tag{72}$$

so that

$$\int_{M_P} |f_{N\alpha}(z)|^2_P \,\mathrm{dv}^r_N(z) = 1.$$

By Propositions 68 and 69, we know that $\lim_{N\to\infty} \|f_{N\alpha}\|^2_\infty$ exists and we have

$$\lim_{N\to\infty} \|f_{N\alpha}\|^2_\infty = c(P,\alpha). \tag{73}$$

Furthermore, by Theorem 1.6, we have

$$\|f_{N\alpha}\|^{2k}_{\mathcal{L}^{2k}(\mathrm{dv}^r_N)} = \left(\tfrac{N}{2\pi}\right)^{-(m+r)(k-1)/2} \|\widehat{\varphi}^P_{N\alpha}\|^{2k}_{2k} = \frac{c(P,\alpha)^{k-1}}{k^{(m+r)/2}}(1+O(N^{-1}))\,. \tag{74}$$

We consider the limit distribution of the sequence of measures

$$\nu_{N,r} := |f_{N\alpha}|^2_* \mathrm{dv}^r_N \tag{75}$$

on the real line. The distribution function $F^r_{N\alpha}(t)$ of the measure $\nu_{N,r}$ is given by

$$F^r_{N\alpha}(t) = \left(\tfrac{N}{2\pi}\right)^{(m+r)/2} \mathrm{Vol}_{M_P}\{z\in M_P\,;\,|\widehat{\varphi}^P_{N\alpha}(z)|^2 > \left(\tfrac{N}{2\pi}\right)^{(m+r)/2} t\}. \tag{76}$$

The distribution function $F^r_{N\alpha}$ defined above can be expressed, in terms of the distribution function $D_{N\alpha}$ for the measure $|\widehat{\varphi}^P_{N\alpha}|^2_* d\mathrm{Vol}_{M_P}$, as

$$F^r_{N\alpha}(t) = \left(\tfrac{N}{2\pi}\right)^{(m+r)/2} D_{N\alpha}\left(\left(\tfrac{N}{2\pi}\right)^{(m+r)/2} t\right).$$

We should note that the total mass of the measure $\nu_{N,r}$ is $\mathrm{Vol}(M_P)(N/2\pi)^{(m+r)/2}$, and hence it tends to infinity as N goes to infinity. But its k-th moment satisfies, for each positive integer k,

$$\int x^k \, d\nu_{N,k}(x) = \frac{c(P,\alpha)^{k-1}}{k^{(m+r)/2}}(1+O_k(N^{-1})) \to \frac{c(P,\alpha)^{k-1}}{k^{(m+r)/2}} \quad (N\to\infty). \tag{77}$$

By Propositions 68 and 69, the support of the measure $\nu_{N,r}$ is contained in a bounded interval in $[0,+\infty)$ which is independent of N.

Furthermore, by (77), the measure $x d\nu_{N,r}(x)$ is a finite measure on the real line whose support lies in a bounded interval in $[0,+\infty)$ which is independent of N. This implies that the sequence of the finite measures $x d\nu_{N,r}(x)$ on the real line has a weak limit, say μ, and it must satisfy

$$\int x^k \, d\mu(x) = \frac{c(P,\alpha)^k}{(k+1)^{(m+r)/2}}, \quad k = 0,1,2,\ldots. \tag{78}$$

Thus, the weak limits of the measures $xd\nu_{N,r}(x)$ are probability measures, and they have supports contained in the interval $[0, c(P,\alpha)]$.

LEMMA **4.1.** *Let c be a positive constant, and let h be a positive integer. Let μ be a probability measure on the real line such that*

$$\int x^k \, d\mu(x) = \frac{c^k}{(k+1)^{h/2}}, \quad k \geq 0.$$

Then μ must coincide with the measure $\rho_{c,h}(x)\, dx$ where

$$\rho_{c,h}(x) = \frac{1}{c\Gamma(h/2)} \chi_{(0,c)}(x) (\log(c/x))^{h/2-1}. \tag{79}$$

Proof. We shall use the following formula:

$$\frac{1}{w^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-wt} t^{s-1} \, dt$$

to compute the Fourier transform

$$\hat{\mu}(\xi) = \int e^{ix\xi} \, d\mu(x)$$

of the measure μ satisfying the condition in the lemma. Substituting $s = h/2$, $w = k$ in the above formula, we get

$$\begin{aligned} \hat{\mu}(\xi) = \sum_{k \geq 0} \frac{(ic\xi)^k}{k!(k+1)^{h/2}} &= \frac{1}{\Gamma(h/2)} \int_0^\infty e^{ic\xi e^{-t}} e^{-t} t^{h/2-1} \, dt \\ &= \frac{1}{c\Gamma(h/2)} \int_0^c e^{i\xi x} (\log(c/x))^{h/2-1} \, dx, \end{aligned}$$

and hence $\hat{\mu}(\xi) = \hat{\rho_{c,h}}(\xi)$. This completes the proof. □

As a corollary to Lemma 4.1, we have the following.

COROLLARY **4.2.** *The sequence of the finite measures $\{xd\nu_{N,r}(x)\}$ converges weakly to the probability measure $\rho_{c(P,\alpha),r}\, dx$, where the density $\rho_{c(P,\alpha),r}$ is given by* (79) *with $c = c(P,\alpha)$ and $h = m + r$.*

Proof of Theorem 1.4. The rescaled distribution function $F^r_{N\alpha}(t)$ for $t > 0$ of the measure $\nu_N = |f_{N\alpha}|^2_* \mathrm{dv}^r_N$ is given by

$$F^r_{N\alpha}(t) = \int \chi_{(t,+\infty)}(x) \, d\nu_{N,r}(x),$$

where $\chi_{(t,+\infty)}$ is the characteristic function of the interval $(t, +\infty)$. Now set $d\mu_{N,r}(x) = xd\nu_{N,r}(x)$ and write

$$F^r_{N\alpha}(t) = \int x^{-1} \chi_{(t,+\infty)}(x) \, d\mu_{N,r}(x).$$

By Corollary 4.2, we know that the sequence of finite measures μ_N converges weakly to the probability measure $\rho_{c(P,\alpha),r}(x)\, dx$. For fixed $t > 0$, by approximating the function

$x^{-1}\chi_{(t,+\infty)}(x)$ by a sequence of continuous functions, and by using the Lebesgue convergence theorem, one easily obtains that, for every $0 < t \le c(P,\alpha)$,

$$\begin{aligned}\lim_{N\to+\infty} F^r_{N\alpha}(t) &= \int x^{-1}\chi_{(t,+\infty)}(x)\rho_{c(P,\alpha),r}(x)\,dx \\ &= \frac{1}{c(P,\alpha)\Gamma((m+r)/2)}\int_t^{c(P,\alpha)} x^{-1}(\log(c(P,\alpha)/x))^{(m+r)/2-1}\,dx \\ &= \frac{1}{c(P,\alpha)\Gamma((m+r)/2)}\int_0^{\log(c(P,\alpha)/t)} s^{(m+r)/2-1}\,ds \\ &= \frac{1}{c(P,\alpha)\Gamma((m+r)/2+1)}(\log(c(P,\alpha)/t))^{(m+r)/2}.\end{aligned}$$

□

4.2. **Non-rescaled distribution functions.** In the previous section, we derived the limit of the rescaled distribution functions $F^r_{N\alpha}(t)$. In the definition of the rescaled function $F^r_{N\alpha}(t)$, the parameter t is rescaled by the factor $(N/2\pi)^{(m+r)/2}$ so that it tends to $+\infty$ as $N \to \infty$. Then the corresponding volume

$$\mathrm{Vol}_{M_P}(z\,;\,|\widehat{\varphi}^P_{N\alpha}(z)|^2 > (N/2\pi)^{(m+r)/2}t) \tag{80}$$

is of order $N^{-(m+r)/2}$. This was the reason why we need to multiply the volume (80) by the extra factor $(N/2\pi)^{(m+r)/2}$ in the definition of the rescaled distribution functions. As a result, the limit distribution has a universal form (Theorem 1.4). However, one may ask, of course, what the limit of the non-rescaled distributions $D_{N\alpha}(t)$ is. But, for each fixed $t > 0$, $D_{N\alpha}(t) \to 0$ as $N \to \infty$ by Theorem 1.2 and Proposition 3.5. Therefore, we need to replace $D_{N\alpha}(t)$ by $D_{N\alpha}(W_N(t))$ for an appropriate sequence $\{W_N(t)\}$ of positive functions in $t > 0$ which compensate for the flattening rate. Our sequence will satisfy the conditions $W_N(t) \to 0$ as $N \to \infty$ and

$$W_N(t) \to 0, \quad W_N(t)^{-1/N} \to W(t) \quad (N \to \infty), \tag{81}$$

for a function $W(t) > 1$. We then have the following limit distribution law:

THEOREM **4.3.** *Let $\{W_N\}$ satisfy* (81). *Then*

$$\lim_{N\to\infty} D_N(W_N(t)) = \mathrm{Vol}_{M_P}((\xi,\rho) \in \mathbb{C}^r \times (\mathbb{R}^*)^{m-r}\,;\,\Psi_\alpha(\xi,\rho) < \log W(t)). \tag{82}$$

In particular, by taking the function W_N as $W_N(t) = e^{-Nt}$, we have

$$\lim_{N\to\infty} D_N(e^{-Nt}) = \frac{1}{\pi^r}\int_{\{(\xi,\rho)\in\mathbb{C}^r\times(\mathbb{R}^*)^{m-r}\,;\,\Psi^P_\alpha(\xi,\rho)<t\}} L(\xi,\rho)\,dm(\xi)d\rho. \tag{83}$$

If α is in the interior P^o of the polytope P, we have

$$\lim_{N\to\infty} D_N(e^{-Nt}) = \int_{\{\rho\in\mathbb{R}^m\,;\,b^P_\alpha(\rho)<t\}} \det A(\rho)\,d\rho. \tag{84}$$

Proof. The proof of (84) is the same as that for (82) and (83), and (83) follows from (82) and Lemma 3.6. Thus, we give a proof only for (82). First of all, we note that the set U_{v_0} introduced in Section 3 is dense in M_P, and hence we may consider the volume of the set

$$S_N(t) := \{(\xi, \zeta) \in \mathbb{C}^r \times (\mathbb{C}^*)^{m-r}\, ;\, |\widehat{\varphi}^P_{N\alpha}(\xi, \zeta)|^2 > W_N(t)\},$$

so that $D_{N\alpha}(W_N(t)) = \mathrm{Vol}_{M_P}(S_N(t))$. Let $(\xi, \zeta) \in S_N$. We write $\zeta = e^{\rho/2+i\theta}$. Then, by Proposition 3.5, we have

$$\Psi_\alpha(\xi, \rho) < \log \left(CN^{-(m+r)/2} W_N(t)(1 + a_N(\xi, \zeta))\right)^{-1/N}, \tag{85}$$

where C is a constant, and a_N is a function of order $O(N^{-1})$. Thus, we obtain

$$\begin{aligned} &D_{N\alpha}(W_N(t)) \\ &\quad= \mathrm{Vol}_{M_P}\left((\xi, \zeta) \in \mathbb{C}^r \times (\mathbb{C}^*)^{m-r}\, ;\, \Psi_\alpha(\xi, \rho) < \log \left(CN^{-(m+r)/2} W_N(t)(1 + a_N(\xi, \zeta))\right)^{-\frac{1}{N}}\right). \end{aligned}$$

This combined with Lemma 3.6 shows the assertion. □

Our final aim is to prove theorem 1.3, which gives the asymptotic limit of the distribution function $D_N(t)$ itself without any rescaling. Since, by Theorem 1.2 and Propositions 3.5 and 3.8, the monomial $|\widehat{\varphi}^P_{N\alpha}|^2$ decays exponentially away from the corresponding invariant torus $\mu_P^{-1}(\alpha)$, it is obvious that $D_N(t)$ tends to zero as $N \to \infty$. So, a problem is to find the decay rate of $D_N(t)$ for any (but fixed) $t > 0$.

For every N and $0 \le r \le m$, we set

$$s_N = \log\left(\tfrac{N}{2\pi}\right)^{(m+r)/2} = \tfrac{m+r}{2} \log\left(\tfrac{N}{2\pi}\right), \quad r_N = \left(\tfrac{s_N}{N}\right)^{1/2}.$$

Lemma 4.4. *Let $t > 0$. Let $\alpha \in P$ be a lattice point with $\dim \mu_P^{-1}(\alpha) = m - r$ and lie in a face F with $\dim F = m - r$. Let $(\xi_N, \zeta_N) \in \mathbb{C}^r \times (\mathbb{C}^*)^{m-r}$ satisfy $|\widehat{\varphi}^P_{N\alpha}(\xi_N, \zeta_N)|^2 > t$. We write $\zeta_N = e^{(\rho^F_\alpha + u_N)/2 + i\theta_N}$. Then, we have*

$$|\xi_N|^2 + |u_N|^2 = O(r_N^2) = O(N^{-1} \log N)$$

locally uniformly in $t > 0$.

Proof. For every $c > 0$, we set

$$B_\alpha(c) = \{(\xi, e^{(\rho^F_\alpha + u)/2} + i\theta) \in \mathbb{C}^r \times (\mathbb{C}^*)^{m-r}\, ;\, |\xi|^2 + |u|^2 < c^2\}. \tag{86}$$

Then, by the argument in the proof of Proposition 3.5, we can find positive constants c_0, C_0 such that

$$c_0(|\xi|^2 + |u|^2) \le \Psi_\alpha(\xi, \rho^F_\alpha + u) \le C_0(|\xi|^2 + |u|^2) \text{ on } B_\alpha(c).$$

Therefore, by Proposition 3.5, we have

$$t \le Ce^{s_N - N\Psi_\alpha(\xi_N, \rho^F_\alpha + u_N)} \le Ce^{s_N - c_0 N(|\xi_N|^2 + |u_N|^2)}$$

with some constant C. Therefore, we obtain

$$|\xi_N|^2 + |u_N|^2 \le \frac{C}{N}(\log(C/t) + s_N),$$

which implies the assertion. □

Thus, to obtain the asymptotic estimate of the distribution function $D_N(t)$, we need to find that of the monomial $|\widehat{\varphi}^P_{N\alpha}|^2$ on the ball $B_\alpha(cr_N)$ of radius $O(r_N)$ around the point $(0,\rho^F_\alpha)$, where $B_\alpha(cr_N)$ is defined in (86).

Lemma 4.5. *Let $c>0$. We denote points $B_\alpha(cr_N)$ as $(\xi,\zeta)=(r_Nw,e^{(\rho^F_\alpha+r_Nu)/2+i\theta})$ with $|w|^2+|u|^2\le c^2$. Then we have*

$$\begin{aligned}&|\widehat{\varphi}^P_{N\alpha}(r_Nw,\rho^F_\alpha+r_Nu)|^2\\ &\qquad =c(P,\alpha)e^{s_N-s_N\langle\, H\Psi_\alpha(0,\rho^F_\alpha)(w,u),(w,u)\,\rangle/2}(1+O(N^{-1/2}(\log N)^{3/2})),\end{aligned}\tag{87}$$

where the Hessian $H\Psi_\alpha(0,\rho^F_\alpha)$ of the function Ψ_α on $\mathbb{C}^r\times\mathbb{R}^{m-r}$ at the point $(0,\rho^F_\alpha)$ is given by (61)

Proof. By a Taylor expansion, we have

$$\Psi_\alpha(r_Nw,\rho^F_\alpha+r_Nu)=\frac{r_N^2}{2}\langle\, H\Psi_\alpha(0,\rho^F_\alpha)(w,u),(w,u)\,\rangle+R(r_N(w,u))$$

for $|w|^2+|u|^2\le c^2$, where $R(r_N(w,u))=O(r_N^3)$. In particular, we have

$$\begin{aligned}e^{-N\Psi_\alpha(r_Nw,\rho^F_\alpha+r_Nu)}&=e^{-s_N\langle\, H\Psi_\alpha(0,\rho^F_\alpha)(w,u),(w,u)\,\rangle/2}(1+O(Nr_N^3))\\ &=e^{-s_N\langle\, H\Psi_\alpha(0,\rho^F_\alpha)(w,u),(w,u)\,\rangle/2}(1+O(N^{-1/2}(\log N)^{3/2})).\end{aligned}$$

¿From this and the estimate in Proposition 3.5, the assertion follows for $1\le r\le m-1$. For $r=0,m$, precisely the same argument replacing Ψ_α by b^P_α or $\log K$ with Theorem 1.2 and Proposition 3.8 will show the assertion. ☐

Proof of Theorem 1.3. For every $t>0$, we set

$$S_N(t)=\{\eta=(\xi,\zeta)\in\mathbb{C}^r\times(\mathbb{C}^*)^{m-r}\,;\,|\widehat{\varphi}^P_{N\alpha}(\xi,\zeta)|^2>t\}.$$

Then, by Lemma 4.4, there is a constant $c>0$ (depending on a fixed $t>0$) such that $S_N(t)\subset B_\alpha(cr_N)$. Let $(\xi,\zeta)=(r_Nw,e^{(\rho^F_\alpha+r_Nu})/2+i\theta)\in B_\alpha(cr_N)$. Then, by Lemma 4.5, $(\xi,\zeta)\in S_N(t)$ if and only if

$$t<c(P,\alpha)e^{s_N-s_N\langle\, H\Psi_\alpha(0,\rho^F_\alpha)(w,u),(w,u)\,\rangle/2}(1+a_N(w,u)),$$

where $a_N(w,u)$ is a function of order $N^{-1/2}(\log N)^{3/2}$ uniformly in (w,u) with $|w|^2+|u|^2\le c^2$, and $c(P,\alpha)$ is define in (70). This is equivalent to the following estimate:

$$\langle\, H\Psi_\alpha(w,u),(w,u)\,\rangle/2<1+\frac{1}{s_N}\log\left(\frac{c(P,\alpha)}{t}(1+a_N(w,u))\right).$$

Therefore, by Lemma 3.6, we have

$$D_{N\alpha}(t)=\frac{1}{\pi^r}\int_{\{(\xi,\rho)=(r_Nw,\rho^P_\alpha+r_Nu)\,;\,\langle\, H(w,u),(w,u)\,\rangle/2<1+\frac{1}{s_N}\log(c(P,\alpha)(1+a(w,u))/t)\}}L(\xi,\rho)\,dm(\xi)d\rho,$$

where we set, for simplicity, $H=H\Psi_\alpha(0,\rho^F_\alpha)$. Changing the variables (ξ,ρ) to (w,u), we have

$$D_{N\alpha}(t)=\frac{r_N^{m+r}}{\pi^r}\int_{\{(w,u)\,;\,\langle\, H(w,u),(w,u)\,\rangle/2<1+\frac{1}{s_N}\log(c(P,\alpha)(1+a_N(w,u))/t)\}}L(r_Nw,\rho^F_\alpha+r_Nu)\,dm(w)du.$$

A Taylor expansion gives

$$L(r_N w, \rho_\alpha^F + r_N u) = L(0, \rho_\alpha^F) + R_N(w,u)$$

with the error term $R_N(w,u) = O(r_N)$ uniformly for $|w|^2 + |u|^2 \le c^2$. Thus, we obtain

$$r_N^{-(m+r)} D_{N\alpha}(t) = \frac{L(0,\rho_\alpha^F)}{\pi^r} \int_{\{(w,u)\,;\,\langle\, H(w,u),(w,u)\,\rangle/2<1+\frac{1}{s_N}\log(c(P,\alpha)(1+a_N(w,u))/t)\}} dm(w)du + O(r_N).$$

Note that $s_N \to \infty$ while $r_N \to 0$ as $N \to \infty$. The function $a_N(w,u)$ tends to zero uniformly in $|w|^2 + |u|^2 \le c^2$. Therefore, we conclude, for a fixed $t > 0$,

$$\begin{aligned}\lim_{N\to\infty} r_N^{-(m+r)} D_{N\alpha}(t) &= \frac{L(0,\rho_\alpha^F)}{\pi^r} \int_{\{(w,u)\in\mathbb{C}^r\times\mathbb{R}^{m-r}\,;\,\langle\, H(w,u),(w,u)\,\rangle/2<1\}} dm(w)du.\\ &= \frac{2^{(m+r)/2}}{\pi^r}\frac{L(0,\rho_\alpha^F)}{\sqrt{\det H}} \mathrm{Vol}_{M_P}(x\in\mathbb{R}^{m+r}\,;\,|x|<1).\end{aligned}$$

By (63) and the well-known formula for the volume of the unit disk in $\mathbb{R}^{m+r}$, we conclude the assertion. □

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